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Some Spectral Approximations of One-Dimensional Fourth-Order Problems

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Abstract

We propose some spectral type collocation methods well suited for the approximation of fourth-order systems. Our model problem is the biharmonic equation, in one dimensional and in two dimensions when the boundary conditions are periodic on one direction. It is proved that the standard Gauss-Lobatto nodes are not the best choice for the collocation points. Then, we propose a new set of nodes related to some generalized Gauss type quadrature formulas. We provide a complete anlaysis of these formulas including some new issues about the asymptotic behaviour of the weights and we apply these results to the analysis of the collocation method.

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I. Introduction.

Spectral methods are well suited for the approximation of the solution of elliptic or parabolic type equations. They are known to be very efficient for second-order problems and their approximation properties are consistant with the infinite order observed through numerical experiments. We refer to the recent book [CHQZ] for a review of most of the current results. However, and quite surprisingly, they have not been much considered for the discretization of fourth-order systems. The regularity of the solution of such problems is not in question, since it is generally higher than for second-order ones. On the other hand, the spaces of discrete functions seem specially appropriate to approximate the solution of high order equations, since they consist either of truncated trigonometric series or of high degree polynomials that are both indefinitely differentiable. Hence, they are contained in Sobolev spaces of any order, which is not the case with finite element spaces. Conforming discretizations can thus be worked out easily. In addition, it can be noted that approximating a linear fourth-order equation with constant coefficients via a Galerkin method using spaces of high degree polynomials with respect to each variable gives optimal results. Only a few collocation methods have been implemented up to now for this kind of problems [O][He], and no numerical analysis has been provided for them (we refer to the review paper [BM3] for a survey of the strategies). We do think that the corner stone of collocation techniques is the choice of the collocation nodes, i.e., the finite set of points in which the equation will be exactly satisfied. In spectral methods, as first suggested by D. GOTTLIEB [Go], these are always built from the nodes of a Gauss type quadrature formula, for two reasons. First, the Lagrange interpolation operator associated with these nodes has very good approximation properties. Second, the quadrature formula allows for writing a variational formulation of the discrete problem; then, the Strang lemma provides an estimate to compare its solution with the exact one. In parallel, if a problem is stated in a variational formulation and not in a strong form, using this quadrature formula leads to consistant discretizations.

For second-order problems, two possibilities arise : choosing the collocation set from the nodes of a Gauss formula or from a Gauss-Lobatto formula. The difference relies on the fact that this set contains some points of the boundary of the domain in the second case and not in the first one. Due to the Dirichlet boundary conditions, the second choice turns out to be more efficient and,

in two dimensions, only Gauss-Lobatto points lead to an optimal approximation error.

In order to discretize fourth-order equations, the question is not so easy to solve since two boundary conditions, one on the function and one on its normal derivative, must be enforced at each boundary node. That is why we propose to use a generalized Gauss type quadrature formula, which approximates the integral of a function on a real interval by a sum of its values at some interior nodes plus its values at the extremities of the interval plus the values of its derivative at the extremities, each of them being multiplied by an appropriate weight. Note that the nodes of the Gauss formula are the zeros of a fixed orthogonal polynomial, those of the Gauss-Lobatto formula are the extrema of this polynomial, i.e., the zeros of its first derivative; by similar arguments, it turns out that the nodes of the generalized formulas must be chosen as the zeros of the successive derivatives of this polynomial. We shall thoroughly study these quadrature formulas, both from theoretical and numerical points of view.

Our aim is of course to discretize fourth-order problems by collocation techniques involving the nodes of the generalized Gauss type formula. We first consider the simple case of a fourth-order equation on a finite real interval, when the solution must vanish at the extremities of the interval, together with its derivative. On this test problem, we compare two discrete problems: in both of them, the exact solution is approximated by a polynomial of the same degree which satisfies the boundary conditions, but the equation is enforced at the interior nodes of either a Gauss-Lobatto formula or an appropriate generalized Gauss type formula. Finally, we consider the equation of the bilaplacian on a rectangle, when the boundary conditions are periodic in one direction and homogeneous in the other one. We discretize this equation by a collocation method using the nodes of the generalized formula in the nonperiodic direction, and we provide a complete numerical analysis of this method. Our theoretical justifications are all given in the generalized framework of weighted Sobolev spaces [BM2], which allows for a simultaneous treatment of the Legendre and Chebyshev collocation techniques (we refer to [CHQZ] for a comparison between them). Our intention is to extend our method to fourth-order problems in a rectangle, provided with inhomogeneous Dirichlet conditions, in a forthcoming paper .

An outline of the paper is as follows. Section II is devoted to the analysis of the generalized quadrature formula. A variational formulation of the monodimensional fourth-order problem for

the bilaplacian in weighted Sobolev spaces is studied in Section III. In Section IV, we compare two collocation techniques for approximating this problem. Finally, in Section V, we extend the method to the two-dimensional equation with mixed periodic-nonperiodic boundary conditions. The paper contains three appendices: the first one gives general approximation properties of high-degree polynomials in the weighted Sobolev spaces of order 2; the second one states the approximation properties of the Lagrange interpolation operator at the nodes of the generalized Gauss type formula; the third one contains several tables of nodes and weights of the quadrature formulas.

II. The generalized quadrature formula.

In all that follows, we denote by Λ the open interval $]-1,1[$. For any integer $n \geqslant 0$, $P_n(\Lambda)$ is the space of the restrictions to $\overline{\Lambda}$ of all polynomials of degree $\leqslant n$.

For any real number $\alpha > -1$, we define the weight ρ_α on Λ by

(II.1) $\quad \forall \zeta \in \Lambda, \quad \rho_\alpha(\zeta) = (1-\zeta^2)^\alpha$.

With this weight, we associate the following scalar product, which is defined on the space of all functions, the square of which is integrable with respect to the measure $\rho_\alpha(\zeta)\,d\zeta$,

(II.2) $\quad (\varphi,\psi)_\alpha = \int_{-1}^{1} \varphi(\zeta)\,\psi(\zeta)\,\rho_\alpha(\zeta)\,d\zeta$.

We recall that a family of orthogonal polynomials with respect to the scalar product $(.,.)_\alpha$ is the family of Jacobi polynomials $(J_n^\alpha)_{n\in\mathbb{N}}$, where J_n^α has degree n and satisfies the condition

(II.3) $$J_n^\alpha(\pm 1) = (\pm 1)^n \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} ,$$

(Γ denotes the classical Euler's gamma-function). A number of properties of these polynomials are well-known (see [DR, §1.13] or [Ho]) : among them, we have

(II.4) $$\forall (m,n) \in \mathbb{N}^2 , \quad \int_{-1}^{1} J_m^\alpha(\zeta)\, J_n^\alpha(\zeta)\, \rho_\alpha(\zeta)\, d\zeta = \delta_{mn} \frac{2^{2\alpha+1}\,\Gamma(n+\alpha+1)^2}{(2n+2\alpha+1)\; n!\; \Gamma(n+2\alpha+1)} ,$$

where δ_{mn} stands for the Kronecker's symbol; the family $(J_n^\alpha)_{n\in\mathbb{N}}$ satisfies the induction formula

(II.5) $$(n+1)(n+2\alpha+1)\, J_{n+1}^\alpha = (2n+2\alpha+1)(n+\alpha+1)\,\zeta\, J_n^\alpha - (n+\alpha)(n+\alpha+1)\, J_{n-1}^\alpha ,$$
$$J_0^\alpha(\zeta) = 1 \quad \text{et} \quad J_1^\alpha(\zeta) = (\alpha+1)\,\zeta$$

We shall also need the following result (see [BM2, Lemma IV.2]), which is valid for $n \geqslant 2$

(II.6) $$\int J_n^\alpha(\zeta)\, d\zeta = \frac{1}{2n+2\alpha+1} \left[\frac{n+2\alpha+1}{n+\alpha+1} J_{n+1}^\alpha - \frac{n+\alpha}{n+2\alpha} J_{n-1}^\alpha \right] ,$$

where $\int J_n^\alpha(\zeta)\, d\zeta$ denotes the primitive function of J_n^α which is orthogonal to 1 for the scalar product $(.,.)_\alpha$.

Moreover, each polynomial J_n^α , $n \in \mathbb{N}$, is an eigenfunction of the operator A_α defined by

(II.7) $\quad A_\alpha v = -\rho_{-\alpha}\,(\rho_{\alpha+1}\, v')'$,

indeed, it satisfies the following ordinary differential equation

(II.8) $\quad (\rho_{\alpha+1}\, J_n^{\alpha\,\prime})' + n(n+2\alpha+1)\,\rho_\alpha\, J_n^\alpha = 0$.

From this equation, we observe a property which is the corner stone of our analysis : the family of

polynomials $(J_n^{\alpha\,\prime})_{n\geq 1}$ is orthogonal with respect to the scalar product $(.,.)_{\alpha+1}$, hence, for any $n \geq 1$, the polynomials $J_n^{\alpha\,\prime}$ and $J_{n-1}^{\alpha+1}$ coincide up to a multiplicative constant. More precisely, using (II.8) and (II.3) to compute $J_n^{\alpha\,\prime}(1)$, we have for any $n \geq 1$

$$(\text{II}.9) \qquad J_n^{\alpha\,\prime} = \frac{n+2\alpha+1}{2}\, J_{n-1}^{\alpha+1} \quad .$$

Now, let N be a fixed integer. In all that follows, we denote by ζ_j^α, $1 \leq j \leq N$, the zeros of the polynomial J_N^α (we drop out the index N for sake of simplicity); it is well-known that these zeros are distinct, so we may assume that : $\zeta_1^\alpha < \zeta_2^\alpha < \ldots < \zeta_N^\alpha$. With each zero ζ_j^α, $1 \leq j \leq N$, we associate a characteristic polynomial Q_j^α, i.e. the only polynomial in $P_{N-1}(\Lambda)$ which is equal to 1 in ζ_j^α and vanishes in ζ_i^α, $1 \leq i \leq N$, $i \neq j$. For any function Φ defined on $\overline{\Lambda}$ and any pair (ρ_-, ρ_+) of real numbers, the notation $\Phi(\pm 1)\,\rho_\pm$ in a summation denotes the sum $\Phi(-1)\,\rho_- + \Phi(+1)\,\rho_+$.

We are interested in quadrature formulas to approximate the integral $\int_{-1}^{1} \Phi(\zeta)\,\rho_\alpha(\zeta)\,d\zeta$, where Φ is a function defined on Λ and is assumed to be smooth; moreover, we want these formulas to be precise, i.e. to be exact on polynomials of the highest possible degree. Two formulas are well-known :

1) the Gauss formula

$$\int_{-1}^{1} \Phi(\zeta)\,\rho_\alpha(\zeta)\,d\zeta \simeq \sum_{j=1}^{N} \Phi(\zeta_j^\alpha)\,\rho_j^{\alpha,G} \quad ;$$

the nodes are the zeros of J_N^α ; for a suitable choice of positive weights $\rho_j^{\alpha,G}$, the formula is exact on $P_{2N-1}(\Lambda)$.

2) the Gauss-Lobatto formula

$$\int_{-1}^{1} \Phi(\zeta)\,\rho_\alpha(\zeta)\,d\zeta \simeq \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+1})\,\rho_j^{\alpha,GL} + \Phi(\pm 1)\,\rho_\pm^{\alpha,GL} \quad ,$$

(with the sommation convention described above); here, the nodes are the zeros of $(1-\zeta^2)\,J_{N+1}^{\alpha\,\prime}$, i.e. by (II.9) the zeros of $J_N^{\alpha+1}$ and the bounds ± 1 of the interval; for a suitable choice of positive weights $\rho_j^{\alpha,GL}$, $1 \leq j \leq N$, and $\rho_\pm^{\alpha,GL}$, the formula is exact on $P_{2N+1}(\Lambda)$.

We propose the following generalized quadrature formulas, depending on a nonnegative integer m :

$$(\text{II}.10) \qquad \int_{-1}^{1} \Phi(\zeta)\,\rho_\alpha(\zeta)\,d\zeta \simeq \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m})\,\rho_j^{\alpha,m} + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(\pm 1)\,\rho_{k,\pm}^{\alpha,m} \quad ,$$

(with the same sommation convention). Here, the interior nodes are the zeros of the polynomial $J_N^{\alpha+m}$. Clearly, with a suitable choice of the weights, the Gauss formula is obtained for m = 0 and

the Gauss-Lobatto formula is obtained for $m = 1$.

Next, we define the weights in order to make the formula as accurate as possible.

Lemma II.1 : *For any real number* $\alpha > -1$ *and for any integer* $m \geq 0$, *there exists a unique* $(N+2m)$*-uple of weights* $\rho_j^{\alpha,m}$, $1 \leq j \leq N$, *and* $\rho_{k,\pm}^{\alpha,m}$, $0 \leq k \leq m-1$, *in* $\mathbb{R}^N \times \mathbb{R}^{2m}$ *such that the quadrature formula* (II.10) *is exact on* $P_{N+2m-1}(\Lambda)$.

Proof : The quadrature formula (II.10) is exact on $P_{N+2m-1}(\Lambda)$ if and only if it is exact on a basis of $P_{N+2m-1}(\Lambda)$, i.e. the vector $(\rho_1^{\alpha,m}, \dots, \rho_N^{\alpha,m}, \rho_{0,\pm}^{\alpha,m}, \dots, \rho_{m-1,\pm}^{\alpha,m})$ is a solution $(\lambda_1, \dots, \lambda_N, \mu_{0,\pm}, \dots, \mu_{m-1,\pm})$ of the linear system

$$\sum_{j=1}^{N} (\zeta_j^{\alpha+m})^n \lambda_j + \sum_{k=0}^{m-1} (d^k(\zeta^n)/d\zeta^k)(\pm 1)\, \mu_{k,\pm} = \int_{-1}^{1} \zeta^n \rho_\alpha(\zeta)\, d\zeta \quad , 0 \leq n \leq N+2m-1 \quad .$$

This is a system of $N+2m$ equations with $N+2m$ unknowns, hence it has a (unique) solution if and only if the only solution of the same system with a zero right-hand member is zero. Therefore, let $(\lambda_1, \dots, \lambda_N, \mu_{0,\pm}, \dots, \mu_{m-1,\pm})$ be a solution of

$$\text{(II.11)} \qquad \forall \Phi \in P_{N+2m-1}(\Lambda), \quad \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m}) \lambda_j \bullet \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(\pm 1)\, \mu_{k,\pm} = 0 \quad .$$

First, choosing $\Phi = (1-\zeta^2)^m Q_i^{\alpha+m}$ in (II.11), we obtain $\lambda_i = 0$, $1 \leq i \leq N$. Next, for ℓ decreasing from $m-1$ to 0, we choose successively $\Phi = (1-\zeta^2)^\ell (1+\zeta) J_N^{\alpha+m}$ and $\Phi = (1-\zeta^2)^\ell (1-\zeta) J_N^{\alpha+m}$ in (II.11), so that we deduce $\mu_{\ell,-} = \mu_{\ell,+} = 0$. That ends the proof.

In all that follows, we assume that the weights $\rho_j^{\alpha,m}$, $1 \leq j \leq N$, and $\rho_{k,\pm}^{\alpha,m}$, $0 \leq k \leq m-1$, are chosen such that the formula (II.10) is exact on $P_{N+2m-1}(\Lambda)$. We derive the

Proposition II.1 : *For any real number* $\alpha > -1$ *and for any integer* $m \geq 0$, *the quadrature formula* (II.10) *is exact on* $P_{2N+2m-1}(\Lambda)$.

Proof : Let Φ be a polynomial in $P_{2N+2m-1}(\Lambda)$. By the Euclidean algorithm, we can find a polynomial Q in $P_{N-1}(\Lambda)$ and a polynomial R in $P_{N+2m-1}(\Lambda)$ such that

$$\Phi = (1-\zeta^2)^m J_N^{\alpha+m} Q + R$$

Next, we compute

$$\int_{-1}^{1} \Phi(\zeta) \rho_\alpha(\zeta)\, d\zeta = \int_{-1}^{1} (1-\zeta^2)^m J_N^{\alpha+m}(\zeta)\, Q(\zeta) \rho_\alpha(\zeta)\, d\zeta + \int_{-1}^{1} R(\zeta) \rho_\alpha(\zeta)\, d\zeta$$

Since $J_N^{\alpha+m}$ is orthogonal to any polynomial of degree $\leq N-1$ with respect to the scalar product $(.,.)_{\alpha+m}$, we obtain

$$\int_{-1}^{1} \Phi(\zeta)\, \rho_\alpha(\zeta)\, d\zeta = \int_{-1}^{1} R(\zeta)\, \rho_\alpha(\zeta)\, d\zeta \quad ,$$

so that, by Lemma II.1,

$$\begin{aligned}\int_{-1}^{1} \Phi(\zeta)\, \rho_\alpha(\zeta)\, d\zeta &= \sum_{j=1}^{N} R(\zeta_j^{\alpha+m})\, \rho_j^{\alpha,m} + \sum_{k=0}^{m-1} (d^kR/d\zeta^k)(\pm 1)\, \rho_{k,\pm}^{\alpha,m} \\ &= \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m})\, \rho_j^{\alpha,m} + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(\pm 1)\, \rho_{k,\pm}^{\alpha,m}\end{aligned}$$

Remark II.1 : Of course, the property of Proposition II.1 is satisfied by the Gauss formula ($m = 0$) and by the Gauss-Lobatto formula ($m = 1$). The idea of building quadrature formulas involving some values of the derivatives of the function is not new (see [DR, § 2.7][St][T]). Note that these formulas are essentially of Gauss type since the choice of the N free nodes on $\mathbb{R}$ that generate the quadrature formula is optimal: any other choice leads to a method which is not exact for all polynomials of $P_{2N+2m-1}(\Lambda)$. However formulas of type (II.10), the nodes of which are zeros of Jacobi polynomials, are specially interesting since they can be used to introduce and justify new spectral collocation methods, as it will appear later.

Since the nodes of formula (II.10) are the zeros of a Jacobi polynomial, they can be computed by the same algorithms as for the Gauss formula [DR, § 2.7] : for instance, the zeros ζ_j^α, $1 \leqslant j \leqslant N$, are the eigenvalues of a tridiagonal symmetric matrix of order N. Details about this method and tables giving the values of these nodes for different values of the parameters α, m and N can be found in Appendix C.

We end this section by giving some properties of the weights $\rho_j^{\alpha,m}$, $1 \leqslant j \leqslant N$, and $\rho_{k,\pm}^{\alpha,m}$, $0 \leqslant k \leqslant m-1$. Our purpose is double : theoretical (for instance, the positivity of some of these weights will be very useful in several proofs) and numerical (since we need them in a number of practical computations).

We begin with the "internal" weights $\rho_j^{\alpha,m}$, $1 \leqslant j \leqslant N$. Let us recall ([DR, § 2.7] or [Sz, formula (3.4.7)]) that the Gauss weights are given by

$$\text{(II.12)} \qquad \rho_j^{\alpha,0} = \Big(\sum_{n=0}^{N-1} \frac{J_n^\alpha(\zeta_j^\alpha)^2}{\int_{-1}^{1} J_n^\alpha(\zeta)^2\, \rho_\alpha(\zeta)\, d\zeta} \Big)^{-1}$$

Due to the Christoffel-Darboux formula [Ho, 12.1][Sz, Thm 3.2.2], this is equivalent to

$$(II.13)\qquad \varrho_j^{\alpha,0} = -\frac{2^{2\alpha+1}\,\Gamma(N+\alpha+1)\,\Gamma(N+\alpha+2)}{(N+1)!\,\Gamma(N+2\alpha+2)}\,\frac{1}{J_{N+1}^{\alpha}(\zeta_j^{\alpha})\,J_N^{\alpha\prime}(\zeta_j^{\alpha})}$$

$$= \frac{2^{2\alpha+1}\,\Gamma(N+\alpha)\,\Gamma(N+\alpha+1)}{N!\,\Gamma(N+2\alpha+1)}\,\frac{1}{J_{N-1}^{\alpha}(\zeta_j^{\alpha})\,J_N^{\alpha\prime}(\zeta_j^{\alpha})}$$

The following lemma shows how to compute the weights $\varrho_j^{\alpha,m}$ from $\varrho_j^{\alpha+m,0}$, $1 \leqslant j \leqslant N$.

Lemma II.2 : *For any real number* $\alpha > -1$ *and for any integer* $m \geqslant 1$, *the weights* $\varrho_j^{\alpha,m}$, $1 \leqslant j \leqslant N$, *satisfy*

$$(II.14)\qquad \varrho_j^{\alpha,m} = (1-(\zeta_j^{\alpha+m})^2)^{-m}\,\varrho_j^{\alpha+m,0}$$

Proof : For any Φ in $P_{N-1}(\Lambda)$, let us compute the integral $\int_{-1}^{1} (1-\zeta^2)^m\,\Phi(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta$ both by formula (II.10) and by the Gauss formula associated with the weight $\varrho_{\alpha+m}$. We obtain

$$\int_{-1}^{1} (1-\zeta^2)^m\,\Phi(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta = \sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^m\,\Phi(\zeta_j^{\alpha+m})\,\varrho_j^{\alpha,m} = \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m})\,\varrho_j^{\alpha+m,0}.$$

Choosing $\Phi = Q_j^{\alpha+m}$, we prove (II.14).

Remark II.2 : Formula (II.14) shows that the weights $\varrho_j^{\alpha,m}$, $1 \leqslant j \leqslant N$, retain some properties of the Gauss weights : indeed, for $1 \leqslant j \leqslant N$, we have

$$(II.15)\qquad \varrho_j^{\alpha,m} > 0 \quad ,$$

$$(II.16)\qquad \varrho_{N+1-j}^{\alpha,m} = \varrho_j^{\alpha,m}$$

Next, we consider the "boundary" weights $\varrho_{k,\pm}^{\alpha,m}$, $0 \leqslant k \leqslant m-1$. Let us first compare $\varrho_{k,+}^{\alpha,m}$ and $\varrho_{k,-}^{\alpha,m}$

Lemma II.3 : *For any real number* $\alpha > -1$ *and for any integer* $m \geqslant 1$, *the weights* $\varrho_{k,\pm}^{\alpha,m}$, $0 \leqslant k \leqslant m-1$, *satisfy*

$$(II.17)\qquad \varrho_{k,+}^{\alpha,m} = (-1)^k\,\varrho_{k,-}^{\alpha,m}$$

Proof : Let Φ be any polynomial in $P_{2N+2m-1}(\Lambda)$. Using (II.10) to compute $\int_{-1}^{1} \Phi(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta = \int_{-1}^{1} \Phi(-\zeta)\,\varrho_\alpha(\zeta)\,d\zeta$, we have

$$\sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m})\,\varrho_j^{\alpha,m} + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(-1)\,\varrho_{k,-}^{\alpha,m} + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(+1)\,\varrho_{k,+}^{\alpha,m}$$

$$= \sum_{j=1}^{N} \Phi(\zeta_{N+1-j}^{\alpha+m})\,\varrho_j^{\alpha,m} + \sum_{k=0}^{m-1} (-1)^k\,(d^k\Phi/d\zeta^k)(+1)\,\varrho_{k,-}^{\alpha,m}$$

$$+ \sum_{k=0}^{m-1} (-1)^k\,(d^k\Phi/d\zeta^k)(-1)\,\varrho_{k,+}^{\alpha,m}$$

whence, by (II.16),

$$\sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(-1)\ \varrho_{k,-}^{\alpha,m} + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(+1)\ \varrho_{k,+}^{\alpha,m}$$
$$= \sum_{k=0}^{m-1} (-1)^k (d^k\Phi/d\zeta^k)(+1)\ \varrho_{k,-}^{\alpha,m} + \sum_{k=0}^{m-1} (-1)^k (d^k\Phi/d\zeta^k)(-1)\ \varrho_{k,+}^{\alpha,m}$$

For ℓ decreasing from $m-1$ to 0, we choose $\Phi(\zeta) = (1-\zeta)^m\,(1+\zeta)^\ell$, which gives (II.17).

The following lemma precises the sign of $\varrho_{0,\pm}^{\alpha,m}$

<u>Lemma II.4</u> : *For any real number* $\alpha > -1$ *and for any integer* $m \geqslant 1$, *the weights* $\varrho_{0,\pm}^{\alpha,m}$ *satisfy*

(II.18) $\quad \varrho_{0,-}^{\alpha,m} = \varrho_{0,+}^{\alpha,m} > 0$

Proof : Applying the formula (II.10) to the function $\Phi = 1$ gives

$$\int_{-1}^{1} \varrho_\alpha(\zeta)\, d\zeta \simeq \sum_{j=1}^{N} \varrho_j^{\alpha,m} + \varrho_{0,\pm}^{\alpha,m} \quad ,$$

so that, by Lemmas II.2 and II.3,

$$2\,\varrho_{0,-}^{\alpha,m} = 2\,\varrho_{0,+}^{\alpha,m} = \int_{-1}^{1} (1-\zeta^2)^{-m}\, \varrho_{\alpha+m}(\zeta)\, d\zeta - \sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^{-m}\, \varrho_j^{\alpha+m,0}$$

We have proved that $\varrho_{0,-}^{\alpha,m} = \varrho_{0,+}^{\alpha,m}$ is equal to the half of the quadrature error of the Gauss formula with N points for the weight $\varrho_{\alpha+m}$, applied to the function $(1-\zeta^2)^{-m}$ Recall the general result giving the quadrature error for the Gauss formula [CM, Th. 2.5] : for any function Φ in $\mathcal{C}^{2N}(\Lambda)$, there exists ξ, $-1 < \xi < 1$, such that

$$\int_{-1}^{1} \Phi(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta - \sum_{j=1}^{N} \Phi(\zeta_j^\alpha)\, \varrho_j^{\alpha,0} = (d^{2N}\Phi/d\zeta^{2N})(\xi)\, \|J_N^\alpha\|_{0,\alpha,\Lambda}^2 \,/\, (2N)!\, k_N^{\alpha\,2} \quad ,$$

where k_N^α denotes the coefficient of ζ^N in J_N^α. Hence, to obtain (II.18), it suffices to prove that the 2N- th derivative of $(1-\zeta^2)^{-m}$ is nonnegative on Λ. Noting that

$$(1-\zeta^2)^{-m} = \Big(\sum_{\ell=0}^{+\infty} \zeta^{2\ell} \Big)^m \quad ,$$

we see that this function and its 2N-th derivative are even and that all the coefficients of their series expansions are nonnegative. Consequently, they are $\geqslant 0$ on Λ.

It is already known [BM2, Lemma V.3] that, for the Gauss-Lobatto formula, the weights $\varrho_{0,\pm}^{\alpha,1}$ are given by

(II.19) $\quad \varrho_{0,-}^{\alpha,1} = \varrho_{0,+}^{\alpha,1} = \int_{-1}^{1} J_N^{\alpha+1}(\zeta)\,(1+\zeta)\, \varrho_\alpha(\zeta)\, d\zeta \,/\, 2\, J_N^{\alpha+1}(1) \quad ,$

which yields by (II.3) and (II.6) (see [BM2, Lemma V.3])

(II.20) $\quad \varrho_{0,-}^{\alpha,1} = \varrho_{0,+}^{\alpha,1} = 2^{2\alpha+1}\, \Gamma(\alpha+1)\, \Gamma(\alpha+2)\, \dfrac{N!}{\Gamma(N+2\alpha+3)}$

The next lemmas show how to compute the weights $\varrho_{k,\pm}^{\alpha,m}$, $0 \leqslant k \leqslant m-1$.

Lemma II.5 : *For any real number* $\alpha > -1$ *and for any integer* $m \geqslant 1$ *, the weights* $\varrho_{k,\pm}^{\alpha,m}$, $1 \leqslant k \leqslant m-1$, *satisfy*

(II.21) $\quad 2(m-1)\, \varrho_{m-1,+}^{\alpha,m} = -\, \varrho_{m-2,+}^{\alpha+1,m-1}$

and

(II.22) $\quad 2k\, \varrho_{k,+}^{\alpha,m} + k(k+1)\, \varrho_{k+1,+}^{\alpha,m} = -\, \varrho_{k-1,+}^{\alpha+1,m-1} \quad , 1 \leqslant k \leqslant m-2$.

Proof : Let us choose $\Phi = (1-\zeta^2)\, \Psi$, where Ψ is an even polynomial in $P_{2N+2m-3}(\Lambda)$. We compute $\int_{-1}^{1} \Phi(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta = \int_{-1}^{1} \Psi(\zeta)\, \varrho_{\alpha+1}(\zeta)\, d\zeta$ by using (II.10) once for α and m, once for $\alpha+1$ and $m-1$. We obtain

$$\sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)\, \Psi(\zeta_j^{\alpha+m})\, \varrho_j^{\alpha,m} + \sum_{k=0}^{m-1} [d^k((1-\zeta^2)\, \Psi)/d\zeta^k](\pm 1)\, \varrho_{k,\pm}^{\alpha,m}$$
$$= \sum_{j=1}^{N} \Psi(\zeta_j^{\alpha+m})\, \varrho_j^{\alpha+1,m-1} + \sum_{k=0}^{m-2} (d^k\Psi/d\zeta^k)(\pm 1)\, \varrho_{k,\pm}^{\alpha+1,m-1}$$

Using (II.14), (II.17) and the fact that Ψ is even, we derive

$$\sum_{k=1}^{m-1} [d^k((1-\zeta^2)\, \Psi)/d\zeta^k](+1)\, \varrho_{k,+}^{\alpha,m} = \sum_{k=0}^{m-2} (d^k\Psi/d\zeta^k)(+1)\, \varrho_{k,+}^{\alpha+1,m-1} \quad .$$

But it turns out that, for $k \geqslant 2$,

$$[d^k((1-\zeta^2)\, \Psi)/d\zeta^k](+1) = -\, 2k\, (d^{k-1}\Psi/d\zeta^{k-1})(+1) - k(k-1)\, d^{k-2}\Psi/d\zeta^{k-2}(+1) \quad ,$$

and that $((1-\zeta^2)\, \Psi)'(+1)$ is equal to $-\, 2\, \Psi(+1)$. Hence, we have

$$-\, 2\, \Psi(+1)\, \varrho_{1,+}^{\alpha,m} - \sum_{k=2}^{m-1} [2k\, (d^{k-1}\Psi/d\zeta^{k-1})(+1) + k(k-1)\, d^{k-2}\Psi/d\zeta^{k-2}(+1)]\, \varrho_{k,+}^{\alpha,m}$$
$$= \sum_{k=0}^{m-2} (d^k\Psi/d\zeta^k)(+1)\, \varrho_{k,+}^{\alpha+1,m-1}$$

For ℓ decreasing from $m-2$ to 0, we choose $\Psi(\zeta) = (1-\zeta^2)^\ell$, which yields (II.21) and (II.22).

Remark II.3 : It follows from formulas (II.21) and (II.22) that the vector of weights $\varrho_{k,+}^{\alpha,m}$, $1 \leqslant k \leqslant m-1$, can be computed from the vector of weights $\varrho_{k,+}^{\alpha+1,m-1}$, $0 \leqslant k \leqslant m-2$, by solving a linear system. The matrix of this system is upper triangular and bidiagonal.

Remark II.4 : Due to formulas (II.17), (II.18), (II.21) and (II.22), we observe that, for $0 \leqslant k \leqslant m-1$,

(II.23) $\quad \varrho_{k,-}^{\alpha,m} = (-1)^k\, \varrho_{k,+}^{\alpha,m} > 0 \quad ,$

hence the only negative weights are the $\varrho_{k,+}^{\alpha,m}$ for odd values of k.

The computation of the weights $\varrho_{0,\pm}^{\alpha,m}$ involves a quite technical result. Let us set, for any $\alpha > -1$ and any integers $N \geq 0$ and $m \geq 1$,

$$(11.24) \qquad X_N^{\alpha,m} = \int_{-1}^{1} (d^m J_{N+m}^{\alpha}/d\zeta^m)(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta \quad .$$

Of course, this quantity is equal to 0 when N is odd. But we are interested first in even values of N.

<u>Lemma II.6</u> : *For any real number* $\alpha > -1$ *and for any integer* $m \geq 1$, *for any nonnegative even integer* N , *the quantity* $X_N^{\alpha,m}$ *is given by*

$$(11.25) \qquad X_N^{\alpha,m} = 2^{2\alpha+m} \frac{\Gamma(\alpha+1)}{(m-1)!} \frac{\Gamma(N+\alpha+m+1)}{\Gamma(N+2\alpha+m+1)} \frac{(N/2+m-1)!}{(N/2)!} \frac{\Gamma(N/2+\alpha+m+1/2)}{\Gamma(N/2+\alpha+3/2)} \quad .$$

Proof : From formula (II.6), we induce

$$J_{N+m}^{\alpha\,\prime} = (2N+2\alpha+2m-1) \frac{N+\alpha+m}{N+2\alpha+m} J_{N+m-1}^{\alpha} + \frac{(N+\alpha+m)(N+\alpha+m-1)}{(N+2\alpha+m)(N+2\alpha+m-1)} J_{N-2+m}^{\alpha\,\prime} \quad .$$

First, derivating this equation $(m-1)$ times, we have at once

$$X_N^{\alpha,m} = (2N+2\alpha+2m-1) \frac{N+\alpha+m}{N+2\alpha+m} X_N^{\alpha,m-1} + \frac{(N+\alpha+m)(N+\alpha+m-1)}{(N+2\alpha+m)(N+2\alpha+m-1)} X_{N-2}^{\alpha,m} \quad .$$

On the other hand, using the same formula with $m = 1$, we obtain

$$X_N^{\alpha,1} = \frac{(N+\alpha+1)(N+\alpha)\ldots(\alpha+2)}{(N+2\alpha+1)(N+2\alpha)\ldots(2\alpha+2)} X_0^{\alpha,1} \quad ,$$

$$= (\Gamma(N+\alpha+2)/\Gamma(\alpha+2))\,(\Gamma(2\alpha+2)/\Gamma(N+2\alpha+2))\,(\alpha+1) \int_{-1}^{1} \varrho_\alpha(\zeta)\, d\zeta$$

which implies by (II.4)

$$X_N^{\alpha,1} = 2^{2\alpha+1}\, \Gamma(\alpha+1)\, \Gamma(N+\alpha+2)/\Gamma(N+2\alpha+2) \quad .$$

Finally, using (II.9), we note that

$$X_0^{\alpha,m} = \int_{-1}^{1} (d^m J_m^{\alpha}/d\zeta^m)(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta = (2\alpha+m+1)(2\alpha+m+2)\ldots(2\alpha+2m) \int_{-1}^{1} \varrho_\alpha(\zeta)\, d\zeta/2^m$$

$$= 2^{2\alpha-m+1}\, (\Gamma(\alpha+1)^2/\Gamma(2\alpha+2))\, (\Gamma(2\alpha+2m+1)/\Gamma(2\alpha+m+1)) \quad .$$

The formula $\Gamma(2s) = (1/\sqrt{2\pi})\, 2^{2s-1/2}\, \Gamma(s)\, \Gamma(s+1/2)$ then implies

$$X_0^{\alpha,m} = 2^{2\alpha+m}\, (\Gamma(\alpha+1)/\Gamma(\alpha+3/2))\, (\Gamma(\alpha+m+1)\Gamma(\alpha+m+1/2)/\Gamma(2\alpha+m+1)) \quad .$$

The induction formula on $X_N^{\alpha,m}$, together with the values of $X_N^{\alpha,1}$ and $X_0^{\alpha,m}$, allows us to prove (II.25). Indeed, to make the computation readable, let us set

$$Y_{N/2}^{\alpha,m} = (\Gamma(N+2\alpha+m+1)/\Gamma(N+\alpha+m+1))\, X_N^{\alpha,m} \,/\, 2^{2\alpha+1}\, \Gamma(\alpha+1) \quad .$$

The sequence $(Y_K^{\alpha,m})_{m \geq 1, K \geq 0}$ satisfies

$$\begin{cases} Y_K^{\alpha,m} = (4K+2\alpha+2m-1)\, Y_K^{\alpha,m-1} + Y_{K-1}^{\alpha,m} \quad , \quad m \geq 2, K \geq 1, \\ Y_K^{\alpha,1} = 1 \quad , \quad K \geq 0, \quad \text{and} \quad Y_0^{\alpha,m} = 2^{m-1}\,\Gamma(\alpha+m+1/2)/\Gamma(\alpha+3/2) \quad , \quad m \geq 1 \quad . \end{cases}$$

Then, it is easy to check (but not so easy to find) the formula, valid for any $m \geq 1$ and $K \geq 0$,

$$Y_K^{\alpha,m} = 2^{m-1} \frac{(K+m-1)!}{K!\,(m-1)!} \frac{\Gamma(K+\alpha+m+1/2)}{\Gamma(K+\alpha+3/2)} \quad .$$

which yields (II.25).

An immediate consequence is the

<u>Corollary II.1</u> : *For any real number* $\alpha > -1$ *, for any nonnegative even integer* N *, the sequence* $(X_N^{\alpha,m})_{m\geq 1}$ *is given by*

(II.26) $$\begin{cases} X_N^{\alpha,m} = (N+2\alpha+m+1)(N+2m-2)(N+2\alpha+3)\, X_N^{\alpha+1,m-1} / 8(\alpha+1)(m-1) \quad , \quad m \geq 2, \\ X_N^{\alpha,1} = 2^{2\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(N+\alpha+2)/\Gamma(N+2\alpha+2) \quad . \end{cases}$$

We also need the

<u>Lemma II.7</u> : *For any real number* $\alpha > -1$ *and for any integer* $k \geq 1$ *, for any nonnegative even integer* N *, the polynomial* J_N^α *satisfies*

(II.27) $$(d^k J_N^\alpha/d\zeta^k)(+1) = ((N+1-k)(N+2\alpha+k)/2(\alpha+k))\,(d^{k-1}J_N^\alpha/d\zeta^{k-1})(+1) \quad .$$

Proof : Formula (II.8) gives

$$(1-\zeta^2)\, J_N^{\alpha\,\prime\prime} - 2(\alpha+1)\,\zeta J_N^{\alpha\,\prime} + N(N+2\alpha+1)\, J_N^\alpha = 0 \quad .$$

Then, it can easily be proven by induction

$$(1-\zeta^2)\,(d^{k+1}J_N^\alpha/d\zeta^{k+1}) - 2(\alpha+k)\,\zeta(d^kJ_N^\alpha/d\zeta^k) + (N+1-k)(N+2\alpha+k)\,(d^{k-1}J_N^\alpha/d\zeta^{k-1}) = 0$$

whence the result.

Formula (II.27) leads us to define, for any $\alpha > -1$ and any integer $N \geq 0$, the sequence $(\lambda_k^\alpha)_{k\geq 0}$ by

(II.28) $$\begin{cases} \lambda_k^\alpha = (N+1-k)(N+2\alpha+k)\,\lambda_{k-1}^\alpha / 2(\alpha+k) \quad , \quad k \geq 1, \\ \lambda_0^\alpha = 1 \quad . \end{cases}$$

<u>Lemma II.8</u> : *For any real number* $\alpha > -1$ *and for any integer* $m \geqslant 2$, *the weights* $\varrho_{0,\pm}^{\alpha,m}$ *are given by*

1) *if* N *is even,*

(II.29) $\varrho_{0,-}^{\alpha,m} = \varrho_{0,+}^{\alpha,m} = Z^{\alpha,m} - \sum_{k=1}^{m-1} \lambda_k^{\alpha+m} \varrho_{k,+}^{\alpha,m}$,

where the sequence $(Z^{\alpha,m})_{m \geqslant 1}$ *is defined by*

$$(\text{II}.30)\quad \left\{ \begin{array}{l} Z^{\alpha,m} = \dfrac{(N+2m-2)(N+2\alpha+3)}{4(\alpha+1)(m-1)} \; Z^{\alpha+1,m-1} \quad , \quad m \geqslant 2, \\ \\ Z^{\alpha,1} = 2^{2\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,N!\,/\Gamma(N+2\alpha+3) \quad ; \end{array} \right.$$

2) *if* N *is odd,*

(II.31) $\varrho_{0,-}^{\alpha,m} = \varrho_{0,+}^{\alpha,m} = T^{\alpha,m} - \sum_{k=1}^{m-2} (\lambda_k^{\alpha+m} + k\,\lambda_{k-1}^{\alpha+m})\, \varrho_{k,+}^{\alpha,m}$

where the sequence $(T^{\alpha,m})_{m \geqslant 1}$ *is defined by*

$$(\text{II}.32)\quad \left\{ \begin{array}{l} T^{\alpha,m} = \dfrac{(N+2m-3)(N+2\alpha+4)}{4(\alpha+1)(m-1)} \; T^{\alpha+1,m-1} \quad , \quad m \geqslant 2, \\ \\ T^{\alpha,1} = 2^{2\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,N!\,/\Gamma(N+2\alpha+3) \quad ; \end{array} \right.$$

Proof : First note that, due to Lemma II.7, we have

$$\lambda_k^{\alpha+m} = (d^{m+k} J_{N+m}^{\alpha}/d\zeta^{m+k})(+1) \,/\, (d^m J_{N+m}^{\alpha}/d\zeta^m)(+1) \quad .$$

1) When N is even, applying formula (II.10) to the polynomial $\Phi = d^m J_{N+m}^{\alpha}/d\zeta^m$ gives at once

$$X_N^{\alpha,m} = 2\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(+1)\,\varrho_{0,+}^{\alpha,m} + 2\sum_{k=1}^{m-1} (d^{m+k} J_{N+m}^{\alpha}/d\zeta^{m+k})(+1)\,\varrho_{k,+}^{\alpha,m}$$

The result follows by setting $Z^{\alpha,m} = X_N^{\alpha,m}/2(d^m J_{N+m}^{\alpha}/d\zeta^m)(+1)$ and using Corollary II.1 and (II.9).

2) When N is odd, we must apply formula (II.10) to the polynomial $\Phi = \zeta\, d^m J_{N+m}^{\alpha}/d\zeta^m$. We obtain

$$\int_{-1}^{1} \zeta\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta = 2\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(+1)\,\varrho_{0,+}^{\alpha,m}$$
$$+\, 2\sum_{k=1}^{m-1} \{(d^{m+k} J_{N+m}^{\alpha}/d\zeta^{m+k})(+1) + k\,(d^{m+k-1} J_{N+m}^{\alpha}/d\zeta^{m+k-1})(+1)\}\,\varrho_{k,+}^{\alpha,m} \quad ,$$

or equivalently

$$\varrho_{0,+}^{\alpha,m} = \int_{-1}^{1} \zeta\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta / 2(d^m J_{N+m}^{\alpha}/d\zeta^m)(+1)$$
$$- \sum_{k=1}^{m-1} (\lambda_k^{\alpha+m} + k\,\lambda_{k-1}^{\alpha+m})\,\varrho_{k,+}^{\alpha,m}$$

It remains to compute

$$\int_{-1}^{1} \zeta\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta = -\int_{-1}^{1} (d^m J_{N+m}^{\alpha}/d\zeta^m)(\zeta)\,\varrho'_{\alpha+1}(\zeta)/2(\alpha+1)\,d\zeta$$
$$= (1/2(\alpha+1)) \int_{-1}^{1} (d^{m+1} J_{N+m}^{\alpha}/d\zeta^{m+1})(\zeta)\,\varrho_{\alpha+1}(\zeta)\,d\zeta$$
$$= ((N+2\alpha+m+1)/4(\alpha+1)) \int_{-1}^{1} (d^m J_{N-1+m}^{\alpha+1}/d\zeta^m)(\zeta)\,\varrho_{\alpha+1}(\zeta)\,d\zeta$$
$$= ((N+2\alpha+m+1)/4(\alpha+1))\, X_{N-1}^{\alpha+1,m} \quad .$$

Setting now $T^{\alpha,m} = (N+2\alpha+m+1)\, X_{N-1}^{\alpha+1,m} / 8\,(\alpha+1)\,(d^m J_{N+m}^{\alpha}/d\zeta^m)(+1)$ and applying once more Corollary II.1 and (II.9), we obtain the result.

<u>Remark II.5</u> : Of course, applying formula (II.10) to the function $\Phi = 1$, one could think to compute the weights $\varrho_{0,\pm}^{\alpha,m}$ by

$$(II.33) \qquad \varrho_{0,-}^{\alpha,m} = \varrho_{0,+}^{\alpha,m} = 2^{2\alpha}\ \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)} - (1/2)\sum_{j=1}^{N} \varrho_j^{\alpha,m} \quad .$$

But, due to the round-off errors, this formula is not so precise as (II.29) and (II.31) for large values of N. As a matter of fact, computing $\varrho_{0,\pm}^{\alpha,m}$ by using the recursive formulas (II.28), (II.30) and (II.32) is also cheaper and easier.

<u>Remark II.6</u> : The computation of the weights from the preceeding formulas seems a little bit complicated. However, we intend to work with low values of m ! Tables giving these weights for different values of the parameters α, m and N can be found in Appendix C.

We explicit here the weights when m is equal to 2, since we will use the corresponding formula in the sequel : the weights $\varrho_j^{\alpha,2}$, $1 \leqslant j \leqslant N$, are given by

$$(II.34) \qquad \varrho_j^{\alpha,2} = \frac{2^{2\alpha+5}\,\Gamma(N+\alpha+2)\,\Gamma(N+\alpha+3)}{N!\,\Gamma(N+2\alpha+5)}\ \frac{1}{(1-(\zeta_j^{\alpha+2})^2)^2\, J_{N-1}^{\alpha+2}(\zeta_j^{\alpha+2})\, J_N^{\alpha+2\,\prime}(\zeta_j^{\alpha+2})} \quad ;$$

the weights $\varrho_{0,\pm}^{\alpha,2}$ and $\varrho_{1,\pm}^{\alpha,2}$ are given respectively by

$$(II.35) \qquad \varrho_{0,-}^{\alpha,2} = \varrho_{0,+}^{\alpha,2} = 2^{2\alpha+2}\ \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+3)}{\alpha+3}\ \frac{N!}{\Gamma(N+2\alpha+5)}\ \{(\alpha+2)\,N^2 + (\alpha+2)(2\alpha+5)\,N + (\alpha+3)(2\alpha+3)\}$$

and

$$(II.36) \qquad \varrho_{1,-}^{\alpha,2} = -\,\varrho_{1,+}^{\alpha,2} = 2^{2\alpha+2}\,\Gamma(\alpha+2)\,\Gamma(\alpha+3)\ \frac{N!}{\Gamma(N+2\alpha+5)} \quad .$$

Thanks to the Stirling's formula, it is an easy matter to note that, when N goes to ∞, the $\varrho_{0,\pm}^{\alpha,m}$ tend to 0 as $N^{-2(1+\alpha)}$ and the $\varrho_{1,\pm}^{\alpha,m}$, $0 \leqslant k \leqslant m-1$, tend to 0 as $N^{-2(2+\alpha)}$.

Let us consider the special (and simpler) case $\alpha = 0$. Then, the weight ϱ_α is equal to 1 and the Jacobi polynomials simply coincide with the Legendre polynomials. The weights $\varrho_{0,\pm}^{0,2}$ and $\varrho_{1,\pm}^{0,2}$ are given by

$$(II.37) \qquad \varrho_{0,-}^{0,2} = \varrho_{0,+}^{0,2} = \frac{8}{3\,(N+1)(N+2)(N+3)(N+4)}\ (2N^2 + 10N + 9)$$

and

$$(II.38) \qquad \varrho_{1,-}^{0,2} = - \varrho_{1,+}^{0,2} = \frac{8}{(N+1)(N+2)(N+3)(N+4)} \quad .$$

Remark II.7 : There exists another standard formula to approximate $\int_{-1}^{1} \Phi(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta$, namely the (left) Gauss-Radau formula

$$\int_{-1}^{1} \Phi(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta \simeq \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha,GR})\, \varrho_j^{\alpha,GR} + \Phi(-1)\, \varrho_-^{\alpha,GR} \quad ,$$

which is not of type (II.10). However, both the Gauss-Radau formula and formula (II.10) are special cases of a last formula that we now introduce.

Let us denote by $(J_n^{\alpha,\beta})_{n \in \mathbb{N}}$ the family of Jacobi polynomials which are orthogonal with respect to the measure $(1+\zeta)^\alpha(1-\zeta)^\beta\, d\zeta$, where $J_n^{\alpha,\beta}$ has degree n. For a fixed integer N, we denote by $\zeta_j^{\alpha,\beta}$, $1 \leqslant j \leqslant N$, the zeros of $J_N^{\alpha,\beta}$. Then, for any integers $m \geqslant 0$ and $p \geqslant 0$, there exists a unique vector of weights $\varrho_j^{\alpha,m,p}$, $1 \leqslant j \leqslant N$, and $\varrho_{k,-}^{\alpha,m,p}$, $0 \leqslant k \leqslant m-1$, and $\varrho_{\ell,+}^{\alpha,m,p}$, $0 \leqslant \ell \leqslant p-1$, in $\mathbb{R}^N \times \mathbb{R}^{m+p}$ such that the quadrature formula

$$(II.39) \qquad \left| \begin{aligned} \int_{-1}^{1} \Phi(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta \simeq & \sum_{j=1}^{N} \Phi(\zeta_j^{\alpha+m,\alpha+p})\, \varrho_j^{\alpha,m,p} \\ & + \sum_{k=0}^{m-1} (d^k\Phi/d\zeta^k)(-1)\, \varrho_{k,-}^{\alpha,m,p} + \sum_{\ell=0}^{p-1} (d^\ell\Phi/d\zeta^\ell)(+1)\, \varrho_{\ell,+}^{\alpha,m,p} \, , \end{aligned} \right.$$

is exact on $\mathbb{P}_{2N+m+p-1}(\Lambda)$. Clearly, the formulas (II.10) coincide with formulas (II.39) in the case $m = p$, while the (left) Gauss-Radau formula is obtained for $m = 1$, $p = 0$.

III. Variational formulation of the Dirichlet problem for the bilaplacian.

We are interested in the approximation of the following model problem : *Find a function* u *defined on* Λ *such that*

$$(\mathrm{III.1})\qquad \begin{cases} u^{(IV)} = f & \text{in } \Lambda \quad , \\ u(\pm 1) = 0 & , \\ u'(\pm 1) = 0 & , \end{cases}$$

where f is a given distribution on Λ. It is well-known that, for any f in $H^{-2}(\Lambda)$, this problem has a unique (variational) solution in $H_0^2(\Lambda)$. However, we need here a more general formulation.

To that aim, for each parameter $\alpha > -1$, we introduce a family of Sobolev spaces associated with the measure $\rho_\alpha(\zeta)\, d\zeta$, where the weight ρ_α has been defined in (II.1). First, we introduce the space

$$(\mathrm{III.2})\qquad L^2_\alpha(\Lambda) = \{ v : \Lambda \to \mathbb{R} \text{ measurable} \, ; \int_{-1}^{1} v^2(\zeta)\, \rho_\alpha(\zeta)\, d\zeta < +\infty \} \quad ;$$

it is a Hilbert space for the scalar product

$$(\mathrm{III.3})\qquad (u,v)_\alpha = \int_{-1}^{1} u(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta \quad ,$$

and we identify it with its dual space; the corresponding norm is noted $\|.\|_{0,\alpha,\Lambda}$. Then, for any integer $k \geqslant 1$, we introduce the Sobolev space

$$(\mathrm{III.4})\qquad H^k_\alpha(\Lambda) = \{ v \in L^2_\alpha(\Lambda) \, ; d^\ell v/d\zeta^\ell \in L^2_\alpha(\Lambda),\, 0 \leqslant \ell \leqslant k \} \quad ;$$

this space is provided with the norm

$$(\mathrm{III.5})\qquad \|v\|_{k,\alpha,\Lambda} = [\int_{-1}^{1} \sum_{\ell=0}^{k} (d^\ell v/d\zeta^\ell)^2\, \rho_\alpha(\zeta)\, d\zeta\,]^{1/2}$$

and with the semi-norm

$$(\mathrm{III.6})\qquad |v|_{k,\alpha,\Lambda} = [\int_{-1}^{1} (d^k v/d\zeta^k)^2\, \rho_\alpha(\zeta)\, d\zeta\,]^{1/2} \quad .$$

For any real number $s \geqslant 0$ which is not an integer, the space $H^s_\alpha(\Lambda)$ is defined by interpolation between $H^{[s]}_\alpha(\Lambda)$ and $H^{[s]+1}_\alpha(\Lambda)$, where [s] is the integral part of s, and its norm is denoted by $\|.\|_{s,\alpha,\Lambda}$. For any nonnegative integer k, we denote by $H^k_{\alpha,0}(\Lambda)$ the closure in $H^k_\alpha(\Lambda)$ of the space $\mathcal{D}(\Lambda)$ of all functions of class $\mathcal{C}^\infty$ having a compact support in Λ; we call $H^{-k}_\alpha(\Lambda)$ its dual space, which is spanned by the derivatives of order k of all functions of $L^2_\alpha(\Lambda)$, and we still use the notation $(.,.)_\alpha$ for the duality pairing between this space and $H^k_{\alpha,0}(\Lambda)$.

A detailed study of the properties of such spaces can be found in [Gr][BM2], we just recall some of them that we need later; assuming that α satisfies $-1 < \alpha < 1$, we have

1) for any $k \geqslant 0$, the mapping : $v \to v\varrho_\alpha$ is an isomorphism from $H^k_{\alpha,0}(\Lambda)$ onto $H^k_{-\alpha,0}(\Lambda)$;

2) for any integer $k \geqslant 1$, the trace mapping : $v \to (v(\pm 1), v'(\pm 1), \dots, (d^{k-1}v/d\zeta^{k-1})(\pm 1))$ is linear continuous from $H^k_\alpha(\Lambda)$ onto $\mathbb{R}^{2k}$, and its kernel is exactly the space $H^k_{\alpha,0}(\Lambda)$;

3) for any integer $k \geqslant 1$, the semi-norm $|.|_{k,\alpha,\Lambda}$ is a norm on $H^k_{\alpha,0}(\Lambda)$, equivalent to $\|.\|_{k,\alpha,\Lambda}$.

Next, we consider the bilinear form a_α, defined on $H^2_\alpha(\Lambda) \times H^2_{\alpha,0}(\Lambda)$ by

(III.7) $\quad a_\alpha(u,v) = \int_{-1}^{1} u''(\zeta)\,(v\varrho_\alpha)''(\zeta)\,d\zeta$.

Clearly, for any f in $H^{-2}_\alpha(\Lambda)$, problem (III.1) is equivalent to the following variational one : *Find* u *in* $H^2_{\alpha,0}(\Lambda)$ *such that*

(III.8) $\quad \forall\, v \in H^2_{\alpha,0}(\Lambda), \quad a_\alpha(u,v) = (f,v)_\alpha$.

In order to study this problem, we need some properties of the form a_α.

The following lemma provides two extensions of the Hardy's inequality that will be useful in the sequel.

<u>Lemma III.1</u> : *For any real number* β, *every function* φ *in* $\mathcal{D}(\Lambda)$ *satisfies*

(III.9) $\quad \int_{-1}^{1} \varphi'^2(\zeta)\,\varrho_\beta(\zeta)\,d\zeta \geqslant (1-2\beta) \int_{-1}^{1} \varphi^2(\zeta)\,\varrho_{\beta-2}(\zeta)\,d\zeta$

and

(III.10) $\quad \int_{-1}^{1} \varphi'^2(\zeta)\,\zeta^2\,\varrho_\beta(\zeta)\,d\zeta \geqslant (1-2\beta) \int_{-1}^{1} \varphi^2(\zeta)\,\zeta^2\,\varrho_{\beta-2}(\zeta)\,d\zeta$.

Proof : The inequality (III.9) follows from

$$\begin{aligned}
0 &\leqslant \int_{-1}^{1} [\varphi'(\zeta)\,\varrho_\beta(\zeta) + (1-2\beta)\,\varphi(\zeta)\,\zeta\,\varrho_{\beta-1}(\zeta)]^2\,\varrho_{-\beta}(\zeta)\,d\zeta \\
&\leqslant \int_{-1}^{1} [\varphi'^2(\zeta)\,\varrho_\beta(\zeta) + (1-2\beta)^2\,\varphi^2(\zeta)\,\zeta^2\,\varrho_{\beta-2}(\zeta) + (1-2\beta)(\varphi^2)'(\zeta)\,\zeta\,\varrho_{\beta-1}(\zeta)]\,d\zeta \\
&\leqslant \int_{-1}^{1} \varphi'^2(\zeta)\,\varrho_\beta(\zeta)\,d\zeta + (1-2\beta) \int_{-1}^{1} \varphi^2(\zeta)\,\varrho_{\beta-2}(\zeta)\,[(1-2\beta)\zeta^2 - (1-\zeta^2) + 2\zeta^2(\beta-1)]\,d\zeta \\
&\leqslant \int_{-1}^{1} \varphi'^2(\zeta)\,\varrho_\beta(\zeta)\,d\zeta - (1-2\beta) \int_{-1}^{1} \varphi^2(\zeta)\,\varrho_{\beta-2}(\zeta)\,d\zeta
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
0 &\leqslant \int_{-1}^{1} [\varphi'(\zeta)\,\zeta\,\varrho_\beta(\zeta) + \varphi(\zeta)\,\varrho_{\beta-1}(\zeta)]^2\,\varrho_{-\beta}(\zeta)\,d\zeta \\
&\leqslant \int_{-1}^{1} [\varphi'^2(\zeta)\,\zeta^2\,\varrho_\beta(\zeta) + \varphi^2(\zeta)\,\varrho_{\beta-2}(\zeta) + (\varphi^2)'(\zeta)\,\zeta\,\varrho_{\beta-1}(\zeta)]\,d\zeta \\
&\leqslant \int_{-1}^{1} \varphi'^2(\zeta)\,\zeta^2\,\varrho_\beta(\zeta)\,d\zeta + \int_{-1}^{1} \varphi^2(\zeta)\,\varrho_{\beta-2}(\zeta)\,[1 - (1-\zeta^2) + 2\zeta^2(\beta-1)]\,d\zeta \\
&\leqslant \int_{-1}^{1} \varphi'^2(\zeta)\,\zeta^2\,\varrho_\beta(\zeta)\,d\zeta - (1-2\beta) \int_{-1}^{1} \varphi^2(\zeta)\,\zeta^2\,\varrho_{\beta-2}(\zeta)\,d\zeta
\end{aligned}$$

<u>Proposition III.1</u> : *Let α satisfy $-1<\alpha<1$. The form a_α is continuous on $H^2_\alpha(\Lambda)\times H^2_{\alpha,0}(\Lambda)$ and elliptic on $H^2_{\alpha,0}(\Lambda)$.*

Proof : The continuity of a_α is an immediate consequence of the inequality

$$|a_\alpha(u,v)| \leqslant \|u''\|_{0,\alpha,\Lambda}\,\|(v\varrho_\alpha)''\|_{0,-\alpha,\Lambda} \leqslant \|u\|_{2,\alpha,\Lambda}\,\|v\varrho_\alpha\|_{2,-\alpha,\Lambda} \quad ,$$

since the mapping : $v \to v\varrho_\alpha$ is continuous from $H^2_{\alpha,0}(\Lambda)$ onto $H^2_{-\alpha,0}(\Lambda)$.

To study the ellipticity, we first note that, for any u in $H^2_{\alpha,0}(\Lambda)$, setting $w = u\varrho_\alpha$, we have

$$a_\alpha(u,u) = a_{-\alpha}(w,w) \quad \text{and} \quad \|u\|_{2,\alpha,\Lambda} \leqslant c\,\|w\|_{2,\alpha,\Lambda} \quad .$$

Hence, it is sufficient to prove the ellipticity of a_α for $\alpha \geqslant 0$. Next, we compute for any α, $-1<\alpha<1$,

$$\begin{aligned} a_\alpha(u,u) &= \int_{-1}^{1} u''^2(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta + 2\int_{-1}^{1} u''(\zeta)u'(\zeta)\,\varrho'_\alpha(\zeta)\,d\zeta + \int_{-1}^{1} u''(\zeta)u(\zeta)\,\varrho''_\alpha(\zeta)\,d\zeta \\ &= \int_{-1}^{1} u''^2(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta - \int_{-1}^{1} u'^2(\zeta)\,\varrho''_\alpha(\zeta)\,d\zeta \\ &\qquad - \int_{-1}^{1} u'^2(\zeta)\,\varrho''_\alpha(\zeta)\,d\zeta - \int_{-1}^{1} u'(\zeta)u(\zeta)\,\varrho^{(III)}_\alpha(\zeta)\,d\zeta \quad , \end{aligned}$$

whence

$$\text{(III.11)} \qquad a_\alpha(u,u) = \int_{-1}^{1} u''^2(\zeta)\,\varrho_\alpha(\zeta)\,d\zeta - 2\int_{-1}^{1} u'^2(\zeta)\,\varrho''_\alpha(\zeta)\,d\zeta + (1/2)\int_{-1}^{1} u^2(\zeta)\,\varrho^{(IV)}_\alpha(\zeta)\,d\zeta.$$

We need

$$\text{(III.12)} \qquad \varrho''_\alpha(\zeta) = \varrho_{\alpha-2}(\zeta)\,(-2\alpha)\,[1+(1-2\alpha)\,\zeta^2] \quad ,$$

$$\text{(III.13)} \qquad \varrho^{(IV)}_\alpha(\zeta) = \varrho_{\alpha-4}(\zeta)\,(-4\alpha)(1-\alpha)\,[3+6(3-2\alpha)\,\zeta^2+(1-2\alpha)(3-2\alpha)\,\zeta^4] \quad .$$

We must study separately two cases : $\alpha \geqslant 1/2$ and $0 \leqslant \alpha \leqslant 1/2$.

1) In the case $\alpha \geqslant 1/2$, we have

$$\begin{aligned} &(1/2)\int_{-1}^{1} u^2(\zeta)\,\varrho^{(IV)}_\alpha(\zeta)\,d\zeta \\ &\quad = -2\alpha(1-\alpha)\int_{-1}^{1} u^2(\zeta)\,\varrho_{\alpha-4}(\zeta)\,[3+6(3-2\alpha)\,\zeta^2+(1-2\alpha)(3-2\alpha)\,\zeta^4]\,d\zeta \\ &\quad \geqslant -2\alpha(1-\alpha)\int_{-1}^{1} u^2(\zeta)\,\varrho_{\alpha-4}(\zeta)\,[3+6(3-2\alpha)\,\zeta^2]\,d\zeta \quad . \end{aligned}$$

Using (III.9) and (III.10) for $\beta = \alpha-2$, together with a density argument, we derive

$$\begin{aligned} &(1/2)\int_{-1}^{1} u^2(\zeta)\,\varrho^{(IV)}_\alpha(\zeta)\,d\zeta \\ &\qquad \geqslant -\,[2\alpha(1-\alpha)/(5-2\alpha)]\int_{-1}^{1} u'^2(\zeta)\,\varrho_{\alpha-2}(\zeta)\,[3+6(3-2\alpha)\,\zeta^2]\,d\zeta \end{aligned}$$

That yields

$$a_\alpha(u,u) \geq \int_{-1}^{1} u''^2(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta + [2\alpha/(5-2\alpha)] \int_{-1}^{1} u'^2(\zeta)\, \varrho_{\alpha-2}(\zeta)$$
$$[2(5-2\alpha) + 2(5-2\alpha)(1-2\alpha)\,\zeta^2 - 3(1-\alpha) - 6(1-\alpha)(3-2\alpha)\,\zeta^2]\, d\zeta$$
$$\geq \int_{-1}^{1} u''^2(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta + [2\alpha/(5-2\alpha)] \int_{-1}^{1} u'^2(\zeta)\, \varrho_{\alpha-2}(\zeta)$$
$$[(7-\alpha) - 2(4-3\alpha+2\alpha^2)\,\zeta^2]\, d\zeta$$

Noting that, for any ζ in Λ and $\alpha \geq 1/2$,

$$(7-\alpha) - 2(4-3\alpha+2\alpha^2)\,\zeta^2 \geq (7-\alpha) - 2(4-3\alpha+2\alpha^2) = -(1-5\alpha+4\alpha^2)$$
$$= -(1-\alpha)(1-4\alpha) \geq 0 \quad ,$$

we finally obtain

$$a_\alpha(u,u) \geq |u|^2_{2,\alpha,\Lambda} \quad .$$

Since $|.|_{2,\alpha,\Lambda}$ is equivalent to the norm $\|.\|_{2,\alpha,\Lambda}$ on $H^2_{\alpha,0}(\Lambda)$, that gives the result.

2) In the case $0 \leq \alpha \leq 1/2$, using again (III.9) and (III.10) for $\beta = \alpha-2$ and a density argument yields

$$a_\alpha(u,u) \geq \int_{-1}^{1} u''^2(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta + 4\alpha(5-2\alpha) \int_{-1}^{1} u^2(\zeta)\, \varrho_{\alpha-4}(\zeta)\, [1 + (1-2\alpha)\,\zeta^2]\, d\zeta$$
$$-2\alpha(1-\alpha) \int_{-1}^{1} u^2(\zeta)\, \varrho_{\alpha-4}(\zeta)\, [3 + 6(3-2\alpha)\,\zeta^2 + (1-2\alpha)(3-2\alpha)\,\zeta^4]\, d\zeta \quad ,$$

that is to say

$$a_\alpha(u,u) \geq \int_{-1}^{1} u''^2(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta + 2\alpha \int_\Lambda u^2(\zeta)\, \varrho_{\alpha-4}(\zeta)$$
$$[(7-\alpha) - 2(4-3\alpha+2\alpha^2)\,\zeta^2 - (1-\alpha)(1-2\alpha)(3-2\alpha)\,\zeta^4]\, d\zeta \quad .$$

We note that, for any ζ in Λ and $\alpha \leq 1/2$,

$$(7-\alpha) - 2(4-3\alpha+2\alpha^2)\,\zeta^2 - (1-\alpha)(1-2\alpha)(3-2\alpha)\,\zeta^4$$
$$\geq -(1-\alpha)(1-4\alpha) - (1-\alpha)(1-2\alpha)(3-2\alpha) = -4(1-\alpha)(1-3\alpha+\alpha^2) \quad ,$$

which proves the ellipticity for $\alpha \geq (3-\sqrt{5})/2$.

In the case $\alpha < (3-\sqrt{5})/2$, we fix a constant λ, $0 < \lambda < 1$, and we use (III.9) for $\beta = \alpha$ then for $\beta = \alpha-2$, again with a density argument, to estimate the first term. We obtain

$$a_\alpha(u,u) \geq (1-\lambda) \int_{-1}^{1} u''^2(\zeta)\, \varrho_\alpha(\zeta)\, d\zeta + \lambda(1-2\alpha)(5-2\alpha) \int_{-1}^{1} u^2(\zeta)\, \varrho_{\alpha-4}(\zeta)\, d\zeta$$
$$-\, 8\alpha(1-\alpha)(1-3\alpha+\alpha^2) \int_{-1}^{1} u^2(\zeta)\, \varrho_{\alpha-4}(\zeta)\, d\zeta \quad .$$

In order to choose

$$\lambda = 8\alpha(1-\alpha)(1-3\alpha+\alpha^2)/(1-2\alpha)(5-2\alpha) \quad ,$$

we just have to check that, if α satisfies $0 \leq \alpha < (3-\sqrt{5})/2$, the right-hand member of the previous line is < 1. This last condition is equivalent to

$$8\alpha(1-\alpha)(1-3\alpha+\alpha^2) < (1-2\alpha)(5-2\alpha) \quad ,$$

i.e. to the positiveness of the polynomial

$$P(\alpha) = 8\alpha^4 - 32\alpha^3 + 36\alpha^2 - 20\alpha + 5 \quad .$$

Computing $P''(\alpha) = 24(4\alpha^2 - 8\alpha + 3) = 24(1-2\alpha)(3-2\alpha)$, we see that, when α goes from 0 to 1/2, P' increases from -20 to -8 and P decreases from 5 to 1/2, hence it is > 1/2. That ends the proof.

Remark III.1 : In the particular case $\alpha = -1/2$ of the Chebyshev weight, the properties of a_α have been proved in [M, Lemma V.1] by a slightly different argument.

An immediate consequence of Proposition III.1 is the

Theorem III.1 : *Let* α *satisfy* $-1 < \alpha < 1$. *For any* f *in* $H_\alpha^{-2}(\Lambda)$, *problem* (III.1) *has a unique solution* u *in* $H^2_{\alpha,0}(\Lambda)$. *Moreover, it satisfies*

(III.14) $\quad \|u\|_{2,\alpha,\Lambda} \leqslant c \, \|f\|_{H_\alpha^{-2}(\Lambda)} \quad .$

We are also in a position to propose a first approximation of problem (III.1). Let N be a given integer. The discrete problem is the following one : *Find a polynomial* u_N *in* $P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ *such that*

(III.15) $\quad \forall \, v_N \in P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad a_\alpha(u_N , v_N) = (f, v_N)_\alpha \quad .$

Theorem III.2 : *Let* α *satisfy* $-1 < \alpha < 1$. *For any* f *in* $H_\alpha^{-2}(\Lambda)$, *problem* (III.15) *has a unique solution* u_N *in* $P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$. *This solution converges to* u *when* N *tends to* $+\infty$. *Moreover, if the solution* u *of problem* (III.1) *belongs to* $H^\sigma_\alpha(\Lambda)$ *for a real number* $\sigma \geqslant 1$, *the following error estimate is satisfied*

(III.16) $\quad \|u-u_N\|_{2,\alpha,\Lambda} \leqslant c \, N^{2-\sigma} \, \|u\|_{\sigma,\alpha,\Lambda} \quad .$

Proof : Proposition III.1 implies at once that problem (III.15) is well-posed; due to (III.8), it also yields that, for any w_N in $P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$,

$$\|u_N-w_N\|^2_{2,\alpha,\Lambda} \leqslant c \, a_\alpha(u_N-w_N \, , u_N-w_N) = c \, a_\alpha(u-w_N \, , u_N-w_N) \leqslant c' \, \|u-w_N\|_{2,\alpha,\Lambda} \, \|u_N-w_N\|_{2,\alpha,\Lambda} ,$$

whence

(III.17) $\quad \|u-u_N\|_{2,\alpha,\Lambda} \leqslant c \inf_{w_N \in P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda)} \|u-w_N\|_{2,\alpha,\Lambda} \quad .$

Choosing $w_N = \pi^{0,2}_{N,2} u$, where $\pi^{0,2}_{N,2}$ is defined in Appendix A, we deduce (III.16) from Theorem A.6.

The convergence is then derived by a standard density argument.

Remark III.2 : By an interpolation argument, one can easily prove that, whenever the data f is in the space $H^{\varrho}_{\alpha}(\Lambda)$ for a real number $\varrho \geqslant 0$, the solution u of (III.1) belongs to $H^{\varrho+4}_{\alpha}(\Lambda)$.

IV. Two collocation methods for the Dirichlet problem for the bilaplacian.

The aim of this section is to analyse and compare two different collocation methods to approximate problem (III.1). The advantage of these methods is that the corresponding mass matrix is diagonal, which is of importance for some algorithmic reasons : the explicit resolution of time-dependent problems or the design of simple preconditioners for instance. It is well-known that, if one wants a spectral collocation algorithm to be accurate, the collocation points must be chosen as the nodes of a quadrature formula. Hence, these methods are related to the quadrature formula (II.10) with respectively m = 1 and m = 2.

Let us introduce some notation. For any integer $m \geq 0$, we consider the bilinear form $(.,.)_{\alpha,m}$ defined on $\mathcal{C}^{m-1}(\overline{\Lambda}) \times \mathcal{C}^{m-1}(\overline{\Lambda})$ by

(IV.1) $(\varphi,\psi)_{\alpha,m} = \sum_{j=1}^{N} \varphi(\zeta_j^{\alpha+m})\, \psi(\zeta_j^{\alpha+m})\, \rho_j^{\alpha,m} + \sum_{k=0}^{m-1} (d^k(\varphi\psi)/d\zeta^k)(\pm 1)\, \rho_{k,\pm}^{\alpha,m}$.

Now, we present the discrete problems. For a function f continuous on Λ, they are the following ones : *Find a polynomial* u_N *in* $P_{N+3}(\Lambda)$ *such that*

$$(IV.2)_m \quad \begin{cases} u_N^{(IV)}(\zeta_j^{\alpha+m}) = f(\zeta_j^{\alpha+m}) & , 1 \leq j \leq N \quad , \\ u_N(\pm 1) = 0 & , \\ u_N'(\pm 1) = 0 & , \end{cases}$$

where m is equal either to 1 or 2. In both problems, the number of equations is equal to the dimension of $P_{N+3}(\Lambda)$.

We begin by studying problem $(IV.2)_2$, which will turn out to be the easier one. We have the

Proposition IV.1 : *Let* α *satisfy* $-1 < \alpha < 1$. *Problem* $(IV.2)_2$ *is equivalent to the following variational one : Find* u_N *in* $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ *such that*

(IV.3) $\forall\, v_N \in P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad a_\alpha(u_N , v_N) = (f,v_N)_{\alpha,2}$.

Proof : By multiplying the first equation in $(IV.2)_2$ by the $(1-\zeta^2)^2\, Q_j^{\alpha+2}$, $1 \leq j \leq N$, which form a basis of $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$, we see that $(IV.2)_2$ is equivalent to find u_N in $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$

such that

$$\forall\, v_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad (u_N^{(IV)}, v_N)_{\alpha,2} = (f, v_N)_{\alpha,2} \quad .$$

Next, since the quadrature formula (II.10) for $m = 2$ is exact on $\mathbb{P}_{2N+3}(\Lambda)$, we observe that $(u_N^{(IV)}, v_N)_{\alpha,2}$ is equal to $(u_N^{(IV)}, v_N)_\alpha$; integrating by parts gives the result.

As in Section III, we obtain the

Theorem IV.1 : *Let* α *satisfy* $-1 < \alpha < 1$. *For any* f *in* $\mathscr{C}^0(\Lambda)$, *problem* $(IV.2)_2$ *has a unique solution* u_N *in* $\mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$.

We want now to estimate the error between u and u_N .

Theorem IV.2 : *Let* α *satisfy* $-1 < \alpha < 1$. *If the solution* u *of problem* (III.1) *belongs to* $H^\sigma_\alpha(\Lambda)$ *for a real number* $\sigma \geq 1$, *and if the data* f *is such that the function* $(1-\zeta^2)^{3/2} f$ *belongs to a space* $H^\rho_\alpha(\Lambda)$ *for a real number* $\rho > 1/2$, *the following error estimate between the solutions of problems* (III.1) *and* $(IV.2)_2$ *is satisfied*

$$\text{(IV.4)} \quad \|u-u_N\|_{2,\alpha,\Lambda} \leq c\,(N^{2-\sigma} \|u\|_{\sigma,\alpha,\Lambda} + N^{1/2-\rho} \|(1-\zeta^2)^{3/2} f\|_{\rho,\alpha,\Lambda}) \quad .$$

Proof : Using Proposition III.1 together with (III.8) and (IV.3) gives, for any w_N in $\mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$,

$$\|u_N-w_N\|^2_{2,\alpha,\Lambda} \leq c\, a_\alpha(u_N-w_N, u_N-w_N) = c\, a_\alpha(u-w_N, u_N-w_N) - (f, u_N-w_N)_\alpha + (f, u_N-w_N)_{\alpha,2} \quad ,$$

whence

$$\text{(IV.5)} \qquad \|u-u_N\|_{2,\alpha,\Lambda} \leq c\,\Big(\inf_{w_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)} \|u-w_N\|_{2,\alpha,\Lambda} + \sup_{v_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)} \frac{(f,v_N)_\alpha - (f,v_N)_{\alpha,2}}{\|v_N\|_{2,\alpha,\Lambda}} \Big) \quad .$$

We choose $w_N = \pi^{0,2}_{N,2} u$; then, we deduce the result from Theorem A.6 and Lemma B.2 of the appendices.

Remark IV.1 : The smoothness assumption we make on f is very weak, since we do not require that f is continuous on $\overline{\Lambda}$, but only on Λ.

Now, we study problem $(IV.2)_1$. Here, we must define the following bilinear form on $\mathcal{C}^4(\overline{\Lambda}) \times \mathcal{C}^0(\overline{\Lambda})$

(IV.6) $\quad a_{\alpha,N}(u,v) = (u^{(IV)},v)_{\alpha,1}$.

This form no longer coincides with the $a_\alpha(.,.)$ on $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$. Nevertheless, we have

Proposition IV.2 : *Let α satisfy $-1 < \alpha < 1$. Problem $(IV.2)_1$ is equivalent to the following variational one : Find u_N in $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ such that*

(IV.7) $\quad \forall\, v_N \in P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad a_{\alpha,N}(u_N , v_N) = (f,v_N)_{\alpha,1}$.

Proof : We obtain (IV.7) simply by multiplying the first equation in $(IV.2)_1$ by the $(1-\zeta^2)^2\, Q_j^{\alpha+1}$, $1 \leqslant j \leqslant N$, which also form a basis of $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$.

Studying the form $a_{\alpha,N}$ requires the following lemma.

Lemma IV.2 : *For any real number $\alpha > -1$, the following inequalities hold*

(IV.8) $\quad c^{-1}\, \| J^\alpha_{N+1} \|^2_{0,\alpha,\Lambda} \leqslant (J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} \leqslant c\, \| J^\alpha_{N+1} \|^2_{0,\alpha,\Lambda}$.

Proof : Let us introduce the following notation : given two quantities $\lambda(N)$ and $\mu(N)$ depending on N, $\lambda(N) \sim \mu(N)$ means that there exists a positive constant c independent of N such that $c^{-1}\lambda(N) \leqslant \mu(N) \leqslant c\,\lambda(N)$. Using the induction formula (II.5) with $n = N+2$, we compute

$$(J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} =$$

$$\frac{(2N+2\alpha+5)(N+\alpha+3)\,(\zeta\, J^\alpha_{N-1} , J^\alpha_{N+2})_{\alpha,1} - (N+\alpha+2)(N+\alpha+3)\,(J^\alpha_{N-1} , J^\alpha_{N+1})_{\alpha,1}}{(N+3)(N+2\alpha+3)} \quad ;$$

but, since $(J^\alpha_{N-1} , J^\alpha_{N+1})_{\alpha,1}$ coincides with $(J^\alpha_{N-1} , J^\alpha_{N+1})_\alpha$ which is equal to 0, we simply have

$$(J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} \sim (\zeta\, J^\alpha_{N-1} , J^\alpha_{N+2})_{\alpha,1} \quad .$$

The same formula (II.5) applied with $n = N-1$ to compute $\zeta\, J^\alpha_{N-1}$ gives

$$(J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} \sim (J^\alpha_N , J^\alpha_{N+2})_{\alpha,1} \quad ,$$

whence, by (II.6),

$$(J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} \sim N^{-1}\, (J^{\alpha\,\prime}_{N+1} , J^\alpha_{N+2})_{\alpha,1} \quad .$$

Since the nodes of formula (II.10) for $m = 1$ are the zeros of $J^{\alpha\,\prime}_{N+1}$, we deduce

$$(J^\alpha_{N-1} , J^\alpha_{N+3})_{\alpha,1} \sim N^{-1}\, \rho^{\alpha,1}_{0,\pm}\, J^{\alpha\,\prime}_{N+1}(\pm 1)\, J^\alpha_{N+2}(\pm 1) \quad .$$

But, due to formula (II.8), we have

$$2(\alpha+1)\, J_n^{\alpha\,\prime}(\pm 1) = n(n+2\alpha+1)\, J_n^\alpha(\pm 1)$$

and, from (II.3) and the Stirling's formula, we derive that, as n goes to $+\infty$, $J_n^\alpha(\pm 1)$ behaves as n^α up to a multiplicative constant. Hence, using (II.20) and once more the Stirling's formula, we obtain

$$(J_{N-1}^\alpha\,,\, J_{N+3}^\alpha)_{\alpha,1} \sim N^{-1}\, N^{-2-2\alpha}\, N^{2+\alpha}\, N^\alpha \sim N^{-1} \quad .$$

On the other hand, it follows from (II.4) and the Stirling's formula that $\|J_{N+1}^\alpha\|_{0,\alpha,\Lambda}^2 \sim N^{-1}$, whence the lemma.

<u>Corollary IV.1</u> : *For any real number* $\alpha > -1$, *the following inequalities hold*

(IV.9) $$c^{-1}\, N^4\, \|J_{N+1}^\alpha\|_{0,\alpha,\Lambda}^2 \leqslant (J_{N+3}^{\alpha\,(IV)}\,,\, J_{N+3}^\alpha)_{\alpha,1} \leqslant c\, N^4\, \|J_{N+1}^\alpha\|_{0,\alpha,\Lambda}^2 \quad .$$

Proof : Derivating four times formula (II.6) with $n = N+2$, we see that

$$J_{N+3}^{\alpha\,(IV)} = \frac{(2N+2\alpha+5)(N+2\alpha+3)}{N+\alpha+3}\, J_{N+2}^{\alpha\,(III)} + \frac{N+\alpha+2}{N+2\alpha+2}\, J_{N+1}^{\alpha\,(IV)} \quad .$$

Repeating this argument three times more, we obtain

$$J_{N+3}^{\alpha\,(IV)} = \lambda_N\, J_{N-1}^\alpha + r_N \quad ,$$

where λ_N is a real coefficient, $\lambda_N \sim N^4$, and r_N is a polynomial with degree $\leqslant N-2$. Then, (IV.9) is a simple consequence of (IV.8).

We can now prove the

<u>Proposition IV.3</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *The form* $a_{\alpha,N}$ *satisfies the following properties of continuity*

(IV.10) $$\forall\, u_N \in P_{N+3}(\Lambda),\ \forall\, v_N \in P_{N+3}(\Lambda) \cap H_{\alpha,0}^2(\Lambda), \quad |a_{\alpha,N}(u_N\,,\, v_N)| \leqslant c\, \|u_N\|_{2,\alpha,\Lambda}\, \|v_N\|_{2,\alpha,\Lambda}$$

and of ellipticity

(IV.11) $$\forall\, u_N \in P_{N+3}(\Lambda) \cap H_{\alpha,0}^2(\Lambda), \quad a_{\alpha,N}(u_N\,,\, u_N) \geqslant c\, \|u_N\|_{2,\alpha,\Lambda}^2 \quad .$$

Proof : Let us write any polynomials u_N and v_N of $P_{N+3}(\Lambda)$ on the form

$$u_N = \sum_{n=0}^{N+3} \hat{u}^n J_n^\alpha \quad \text{and} \quad v_N = \sum_{n=0}^{N+3} \hat{v}^n J_n^\alpha \quad .$$

Since the quadrature formula (II.10) with m = 1 is exact on $P_{2N+1}(\Lambda)$, we derive from (III.7) and (IV.6)

$$a_{\alpha,N}(u_N\,,\, v_N) - a_\alpha(u_N\,,\, v_N) = \hat{u}^{N+3}\, \hat{v}^{N+3}\, (J_{N+3}^{\alpha\,(IV)}\,,\, J_{N+3}^\alpha)_{\alpha,1} \quad .$$

This formula, together with Proposition III.1 and the positivity of $(J_{N+3}^{\alpha\,(IV)}, J_{N+3}^{\alpha})_{\alpha,1}$ proved in Corollary IV.1, yields the property of ellipticity (IV.11). On the other hand, in order to prove (IV.10), it suffices to check that for any v_N in $P_{N+3}(\Lambda)$,

(IV.12) $\quad (\hat{v}^{N+3})^2\,(J_{N+3}^{\alpha\,(IV)}, J_{N+3}^{\alpha})_{\alpha,1} \leq c\,|v_N|^2_{2,\alpha,\Lambda}$.

Writing now $v_N'' = \sum_{n=0}^{N+1} \hat{z}^n J_n^{\alpha}$, we have

$$|v_N|^2_{2,\alpha,\Lambda} \simeq \sum_{n=0}^{N+1} (\hat{z}^n)^2\,\|J_n^{\alpha}\|^2_{0,\alpha,\Lambda} \quad ;$$

using (II.6), we compute

$$J_{N+3}^{\alpha\,''} = (2N+2\alpha+3)(2N+2\alpha+5)\,\frac{(N+\alpha+2)(N+\alpha+3)}{(N+2\alpha+2)(N+2\alpha+3)}\,J_{N+1}^{\alpha} + q_N \quad ,$$

where the degree of q_N is $\leq N$; hence, comparing the two expansions of v_N'' yields

$$\hat{z}^{N+1} = (2N+2\alpha+3)(2N+2\alpha+5)\,\frac{(N+\alpha+2)(N+\alpha+3)}{(N+2\alpha+2)(N+2\alpha+3)}\,\hat{v}^{N+3} \quad ,$$

so that

$$|v_N|^2_{2,\alpha,\Lambda} \geq c\,N^4\,(\hat{v}^{N+3})^2\,\|J_{N+1}^{\alpha}\|^2_{0,\alpha,\Lambda} \quad .$$

This inequality, together with Corollary IV.1, implies (IV.12), hence the proposition.

As previously, we derive that the discrete problem is well-posed.

<u>Theorem IV.3</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any* f *in* $\mathcal{C}^0(\Lambda)$, *problem* $(IV.2)_1$ *has a unique solution* u_N *in* $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$.

The error between u and u_N is given in the

<u>Theorem IV.4</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *If the solution* u *of problem* (III.1) *belongs to* $H^{\sigma}_{\alpha}(\Lambda)$ *for a real number* $\sigma \geq 1$, *and if the data* f *is such that the function* $(1-\zeta^2)\,f$ *belongs to a space* $H^{\rho}_{\alpha}(\Lambda)$ *for a real number* $\rho > 1/2$, *the following error estimate between the solutions of problems* (III.1) *and* $(IV.2)_1$ *is satisfied*

(IV.13) $\quad \|u-u_N\|_{2,\alpha,\Lambda} \leq c\,(\,N^{2-\sigma}\,\|u\|_{\sigma,\alpha,\Lambda} + N^{1/2-\rho}\,\|(1-\zeta^2)\,f\|_{\rho,\alpha,\Lambda}\,)$

Proof : Using Proposition IV.3 together with (III.8) and (IV.7) gives, for any w_N in $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$,

$$\|u_N-w_N\|^2_{2,\alpha,\Lambda} \leq c\,a_{\alpha,N}(u_N-w_N\,,\,u_N-w_N) = c\,a_{\alpha,N}(u-w_N\,,\,u_N-w_N) - (f,u_N-w_N)_{\alpha} + (f,u_N-w_N)_{\alpha,1}$$

so that

$$(\mathrm{IV}.14)\qquad \|u-u_N\|_{2,\alpha,\Lambda} \leqslant c\,\Big(\inf_{w_N \in \mathbb{P}_{N+3}(\Lambda)\cap H^2_{\alpha,0}(\Lambda)} \|u-w_N\|_{2,\alpha,\Lambda} + \sup_{v_N \in \mathbb{P}_{N+3}(\Lambda)\cap H^2_{\alpha,0}(\Lambda)} \frac{(f,v_N)_\alpha - (f,v_N)_{\alpha,1}}{\|v_N\|_{2,\alpha,\Lambda}} \Big) .$$

We choose $w_N = \pi^{0,2}_{N,2}u$; then, we deduce the result from Theorem A.6 and Lemma B.2 of the appendices.

Remark IV.2 : Here again, the continuity of the data f on $\bar{\Lambda}$ is not necessary for the discrete solution to converge to the exact one. However, the assumption of Theorem IV.4 is stronger that the assumption of Theorem IV.2 for the same order of accuracy : indeed, taking $f(\zeta) = (1-\zeta^2)^\beta$, we know that $(1-\zeta^2)^{3/2} f$ belongs to $H^\rho_\alpha(\Lambda)$ if and only if ρ is $< \beta + (4+\alpha)/2$, while $(1-\zeta^2) f$ belongs to $H^\rho_\alpha(\Lambda)$ if and only if ρ is $< \beta + (3+\alpha)/2$.

Remark IV.3 : By applying a standard duality method, one could obtain an error bound in the norm $\|.\|_{0,\alpha,\Lambda}$. Precisely, under the assumptions of Theorem IV.2 or IV.4, it would be possible to prove the following estimate between the solution u of problem (III.1) and the solution u_N of problem $(\mathrm{IV}.2)_m$

$$(\mathrm{IV}.15)\qquad \|u-u_N\|_{0,\alpha,\Lambda} \leqslant c\,(N^{-\sigma} \|u\|_{\sigma,\alpha,\Lambda} + N^{1/2-\rho} \|(1-\zeta^2)^{(m+1)/2} f\|_{\rho,\alpha,\Lambda}) .$$

However, the term $N^{1/2-\rho} \|(1-\zeta^2)^{(m+1)/2} f\|_{\rho,\alpha,\Lambda}$ is not improved; hence, this last estimate is not of great interest, since this term is the worst one (indeed, the fact that $(1-\zeta^2)^m f$ belongs to $H^\rho_\alpha(\Lambda)$ implies that f is in $H^{\rho-m}_\alpha(\Lambda)$, hence that u belongs to $H^{\rho+4-m}_\alpha(\Lambda)$, and the other term in (IV.4) or (IV.13) is $N^{m-2-\rho} \|u\|_{\rho+4-m,\alpha,\Lambda}$, which is smaller).

Remark IV.4 : Several other collocation methods seem natural to approximate problem (III.1). Let us consider two of them and prove that they are not so good as problems $(\mathrm{IV}.2)_1$ and $(\mathrm{IV}.2)_2$.

First, an immediate analogue of these problems consists of using the Gauss points, which gives : *Find a polynomial* u_N *in* $\mathbb{P}_{N+3}(\Lambda)$ *such that*

$$(\mathrm{IV}.16)\qquad \begin{cases} u_N^{(IV)}(\zeta_j^\alpha) = f(\zeta_j^\alpha) & , 1 \leqslant j \leqslant N , \\ u_N(\pm 1) = 0 , & \\ u_N'(\pm 1) = 0 . & \end{cases}$$

Of course, setting now

$$\tilde{a}_{\alpha,N}(u,v) = (u^{(IV)},v)_{\alpha,0} ,$$

we see that this problem is equivalent to the following variational one : *Find* u_N *in* $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ *such that*

$$\forall\, v_N \in P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad \tilde{a}_{\alpha,N}(u_N, v_N) = (f, v_N)_{\alpha,0} \quad .$$

However, choosing $v = (1-\zeta^2)^2 J_N^{\alpha\,''}$, we observe that

$$\tilde{a}_{\alpha,N}(v,v) \leqslant c\,N^6 \quad \text{and} \quad \|v\|_{2,\alpha,\Lambda} \geqslant c'\,N^7 \quad ,$$

which shows that the constant of ellipticity of the form $\tilde{a}_{\alpha,N}$ is not bounded from below independently of N. Consequently that the approximation of u by u_N is not optimal.

Another method consists of searching u_N as a polynomial of lower degree, which is achieved in the following : *Find a polynomial* u_N *in* $P_{N+1}(\Lambda)$ *such that*

$$\text{(IV.17)} \qquad \begin{cases} u_N^{(IV)}(\zeta_j^{\alpha+m}) = f(\zeta_j^{\alpha+m}) & , 2 \leqslant j \leqslant N-1 \quad , \\ u_N(\pm 1) = 0 & , \\ u_N'(\pm 1) = 0 & , \end{cases}$$

where m can be equal to 0, 1 or 2. This problem admits the variational formulation : *Find* u_N *in* $P_{N+1}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ *such that*

$$\forall\, v_N \in P_{N+1}(\Lambda) \cap H^1_{\alpha,0}(\Lambda)\; /\; v_N(\zeta_1^{\alpha+m}) = v_N(\zeta_N^{\alpha+m}) = 0, \quad (u_N^{(IV)}, v_N)_\alpha = (f, v_N)_{\alpha,m} \quad .$$

But, in this case, the norm of the form : $(u,v) \to (u^{(IV)}, v)_\alpha$ on the discrete spaces is no longer bounded independently of N, which prevents the discrete problem to be well-posed (indeed, in the simplest case $\alpha = 0$, denoting by k the integral part of $(N-1)/2$, we set

$$u_N(\zeta) = \int_{-1}^{\zeta} (J^0_{2k+1} - J^0_{2k-1})(\xi)\, d\xi \quad \text{and} \quad v_N(\zeta) = (1-\zeta^2)((\zeta_1^m)^2 - \zeta^2) \quad ;$$

it is easy to check that

$$|(u_N^{(IV)}, v_N)_0| \geqslant c\,N \quad \text{and} \quad \|u_N\|_{2,0,\Lambda}\, \|v_N\|_{2,0,\Lambda} \leqslant c\,N^{1/2} \quad ,$$

so that the norm is $\geqslant c\,N^{1/2}$).

V. A collocation method for the periodic nonperiodic Dirichlet problem for the bilaplacian.

We complete the analysis of the best collocation method described in Section IV (collocation at the N nodes of the generalized Gauss quadrature formula for $m = 2$) by applying it to a bidimensional Dirichlet problem for the bilaplacian, when the boundary conditions are periodic in the first direction and homogeneous in the second one. More precisely, let Ω denote the domain $\Theta \times \Lambda$, where Θ and Λ stand respectively for the intervals $]-\pi,\pi[$ and $]-1,1[$. The generic point in Ω is denoted by $\mathbf{x} = (x,y)$.

For a given distribution f on Ω, the problem we want to approximate is the following one :
Find a function u *defined on* Ω *such that*

$$
\text{(V.1)} \quad \left|\begin{array}{ll}
\Delta^2 u = f \quad \text{in } \Omega \quad , & \\
u(-\pi,y) = u(+\pi,y) \quad \text{and} \quad u(x,\pm 1) = 0 & , \mathbf{x} = (x,y) \in \overline{\Omega} \quad , \\
(\partial u/\partial x)(-\pi,y) = (\partial u/\partial x)(+\pi,y) \quad \text{and} \quad (\partial u/\partial y)(x,\pm 1) = 0 & , \mathbf{x} = (x,y) \in \overline{\Omega} \quad .
\end{array}\right.
$$

The boundary conditions are periodic in the x-direction and homogeneous Dirichlet in the y-direction.

<u>Example V.1</u> : An example of problem (V.1) is given by the stationary Stokes equations governing the flow of a viscous incompressible fluid between two parallel planes $y = \pm 1$, when the body forces are parallel to the plane spanned by the x and y-directions and depends only on the two coordinates x and y :

$$
\text{(V.2)} \quad \left|\begin{array}{l}
-\nu\, \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \quad , \\
\operatorname{div} \mathbf{u} = 0 \quad .
\end{array}\right.
$$

Here, **u** represents the velocity of the fluid, p is the pressure and the kinematic viscosity ν is a fixed parameter > 0. A model of this problem is obtained by reducing it to the previously defined domain Ω and providing it with the boundary conditions

$$
\text{(V.3)} \qquad \mathbf{u}(-\pi,y) = \mathbf{u}(+\pi,y) \quad \text{and} \quad \mathbf{u}(x,\pm 1) = 0 \quad , \mathbf{x} = (x,y) \in \overline{\Omega} \quad .
$$

Then, following the techniques of [BMM, Prop. II.2], we can prove that, for any **u** in $H^\sigma(\Omega)$, $\sigma \geq 1$, there exists a unique stream function u in $H^{\sigma+1}(\Omega)$ such that

$$
\text{(V.4)} \qquad \mathbf{u} = \mathbf{curl}\, u \quad \text{in } \Omega \quad .
$$

This function u is a solution of problem (V.1) with $f = \mathrm{curl}\, \mathbf{f}$. Similar results can be obtained in weighted Sobolev spaces [MM].

The analysis of problem (V.1) requires some notation. First, if X is a Banach space, for any real number $s \geqslant 0$, we denote by $H^s_{\#}(\Theta,X)$ the closure in $H^s(\Theta,X)$ of the space of all functions of class $\mathcal{C}^\infty$ on $\overline{\Theta}$ which are periodic with a 2π-period, and by $H^{-s}_{\#}(\Theta,X')$ its dual space. Let $\hat{v}^\ell$, $\ell \in \mathbb{Z}$, stand for the Fourier coefficients of any function v in $L^2(\Theta,X)$, we recall that the mapping:

$$v \to \left(\sum_{\ell\in\mathbb{Z}} (1+\ell^2)^s \|\hat{v}^\ell\|^2_X \right)^{1/2}$$

is a norm on $H^s_{\#}(\Theta,X)$, equivalent to that induced by $H^s(\Theta,X)$. Then, defining the Fourier coefficients $\hat{f}^\ell$, $\ell \in \mathbb{Z}$, of any distribution f in $H^{-s}_{\#}(\Theta,X')$, we see that the norm on $H^{-s}_{\#}(\Theta,X')$ is given by

$$\|f\|_{H^{-s}_{\#}(\Theta,X')} = \left(\sum_{\ell\in\mathbb{Z}} (1+\ell^2)^{-s} \|\hat{f}^\ell\|^2_{X'} \right)^{1/2} .$$

Next, for each parameter $\alpha > -1$, for any real number $s \geqslant 0$, we introduce the space

(V.5) $$H^s_{\alpha,\#}(\Omega) = H^s_{\#}(\Theta,L^2_\alpha(\Lambda)) \cap L^2(\Theta,H^s_\alpha(\Lambda)) ,$$

and provide it with the norm

(V.6) $$\|v\|_{s,\alpha,\#,\Omega} = \left\{ \sum_{\ell\in\mathbb{Z}} \left(\ell^{2s} \|\hat{v}^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{v}^\ell\|^2_{s,\alpha,\Lambda} \right) \right\}^{1/2}$$

For any integer $k \geqslant 0$, we also consider the space

(V.7) $$H^k_{\alpha,\#0}(\Omega) = H^k_{\#}(\Theta,L^2(\Lambda)) \cap L^2(\Theta, H^k_{\alpha,0}(\Lambda)) ,$$

and we denote its dual space by $H^{-k}_{\alpha,\#}(\Omega)$.

Clearly, if the distribution f is given in $H^{-2}_{\alpha,\#}(\Omega)$, problem (V.1) is equivalent to the following variational one: *Find* u *in* $H^2_{\alpha,\#0}(\Omega)$ *such that*

(V.8) $$\forall\, v \in H^2_{\alpha,\#0}(\Omega),\quad \int_{-\pi}^{\pi} (\Delta^2 u,v)_\alpha\, dx = \int_{-\pi}^{\pi} (f,v)_\alpha\, dx .$$

Finally, taking the Fourier transform of this equation gives the equivalent formulation : *Find* u *in* $H^2_{\alpha,\#0}(\Omega)$ *such that, for any integer* ℓ ,

(V.9) $$\forall\, z \in H^2_{\alpha,0}(\Lambda),\quad (\ell^4\hat{u}^\ell - 2\,\ell^2\,\hat{u}^{\ell\,\prime\prime} + \hat{u}^{\ell\,(IV)}, z)_\alpha = (\hat{f}^\ell, z)_\alpha .$$

Let us define the bilinear form on $H^2_{\alpha,0}(\Lambda) \times H^2_{\alpha,0}(\Lambda)$

(V.10) $$b^\ell_\alpha(w,z) = \int_{-1}^{1} \{ \ell^4\, w(y)\, z(y)\, \rho_\alpha(y) + 2\,\ell^2\, w'(y)\, (z\rho_\alpha)'(y) + w''(y)\, (z\rho_\alpha)''(y) \}\, dy$$

Proposition V.1 : *Let α satisfy $-1 < \alpha < 1$. For any integer ℓ, the form b_α^ℓ satisfies the following properties of continuity*

(V.11) $$\begin{cases} \forall\, w \in H^2_{\alpha,0}(\Lambda),\ \forall\, z \in H^2_{\alpha,0}(\Lambda), \\ |b_\alpha^\ell(w,z)| \leq c\,(\,\ell^4 \|w\|^2_{0,\alpha,\Lambda} + \|w\|^2_{2,\alpha,\Lambda}\,)^{1/2}\ (\,\ell^4 \|z\|^2_{0,\alpha,\Lambda} + \|z\|^2_{2,\alpha,\Lambda}\,)^{1/2} \end{cases},$$

and of ellipticity

(V.12) $$\forall\, w \in H^2_{\alpha,0}(\Lambda),\quad b_\alpha^\ell(w,w) \geq c\,(\,\ell^4 \|w\|^2_{0,\alpha,\Lambda} + \|w\|^2_{2,\alpha,\Lambda}\,)$$

Proof : Integrating by parts, we have for any w and z in $H^2_{\alpha,0}(\Lambda)$,

$$b_\alpha^\ell(w,z) = \int_{-1}^{1} \{\,\ell^4 w(y)\, z(y)\, \varrho_\alpha(y) - 2\,\ell^2 w(y)\, (z\varrho_\alpha)''(y) + w''(y)\, (z\varrho_\alpha)''(y)\,\}\, dy \quad ,$$

so that

$$|b_\alpha^\ell(w,z)| \leq \ell^4 \|w\|_{0,\alpha,\Lambda} \|z\|_{0,\alpha,\Lambda} + 2\,\ell^2 \|w\|_{0,\alpha,\Lambda} \|z\varrho_\alpha\|_{2,-\alpha,\Lambda} + \|w\|_{2,\alpha,\Omega} \|z\varrho_\alpha\|_{2,\alpha,\Lambda} \quad .$$

Using the continuity of the mapping : $v \to v\varrho_\alpha$ from $H^2_{\alpha,0}(\Lambda)$ onto $H^2_{-\alpha,0}(\Lambda)$, we obtain (V.11). Moreover, for $z = w$, we write

$$b_\alpha^\ell(w,z) = \int_{-1}^{1} \{\,\ell^4 w^2(y)\, \varrho_\alpha(y) + 2\,\ell^2 w'(y)\, (w\varrho_\alpha)'(y) + w''(y)\, (w\varrho_\alpha)''(y)\,\}\, dy \quad ,$$

and (V.12) follows from Proposition III.1 and the ellipticity [BM2, Lemma III.4] of the form : $(w,z) \to \int_{-1}^{1} w'(y)\, (z\varrho_\alpha)'(y)\, dy$ on $H^1_{\alpha,0}(\Lambda)$.

Theorem V.1 : *Let α satisfy $-1 < \alpha < 1$. For any f in $H^{-2}_{\alpha,\#}(\Omega)$, problem (V.1) has a unique solution u in $H^2_{\alpha,\#0}(\Omega)$. Moreover, it satisfies*

(V.13) $$\|u\|_{2,\alpha,\#,\Omega} \leq c\, \|f\|_{H^{-2}_{\alpha,\#}(\Omega)} \quad .$$

Proof : Any distribution f in $H^{-2}_{\alpha,\#}(\Omega)$ can be written $f = f_\# + f_0$, where $f_\#$ belongs to $H^{-2}_{\#}(\Theta, L^2_\alpha(\Lambda))$ and f_0 belongs to $L^2(\Theta, H^{-2}_\alpha(\Lambda))$, with

$$\|f\|_{H^{-2}_{\alpha,\#}(\Omega)} \leq \|f_\#\|_{H^{-2}_\#(\Theta, L^2_\alpha(\Lambda))} + \|f_0\|_{L^2(\Theta, H^{-2}_\alpha(\Lambda))} \quad .$$

Due to Proposition V.1, for any integer ℓ, equation (V.9) has a unique solution w^ℓ satisfying

$$\ell^4 \|w^\ell\|^2_{0,\alpha,\Lambda} + \|w^\ell\|^2_{2,\alpha,\Lambda} \leq c\,(\,(1+\ell^2)^{-2}\, \|\hat f^\ell_\#\|^2_{0,\alpha,\Lambda} + \|\hat f^\ell_0\|^2_{H^{-2}_\alpha(\Lambda)}\,) \quad .$$

The function u, the Fourier coefficients of which are the w^ℓ, $\ell \in \mathbb{Z}$, is the unique solution of (V.1) in $H^2_{\alpha,\#0}(\Omega)$ and satisfies (V.13).

In order to approximate problem (V.1), we fix an integer $L \geq 1$. Then, for any Banach space

X, we consider the space $S_L(\Theta,X)$ of trigonometric series of order $\leqslant L$ on Θ, i.e.,

(V.14) $\quad S_L(\Theta,X) = \{ v_L = \sum_{\ell=-L}^{L} \hat{v}^\ell \exp(i\ell x), \hat{v}^\ell \in X \}$.

Next, we introduce the nodes

(V.15) $\quad x_k = 2k\pi/(2L+1) \quad , \quad -L \leqslant k \leqslant L$.

Let i_N^* denote the interpolation operator at the nodes x_k , $-L \leqslant k \leqslant L$: for any function v in $\mathcal{C}^0(\Theta,X)$, $i_L^* v$ belongs to $S_K(\Theta,X)$ and satisfies

(V.16) $\quad (i_L^* v)(x_k) = v(x_k) \quad , \quad -L \leqslant k \leqslant L$.

Now, let δ denote the discretization parameter (L,N), where L is $\geqslant 1$ and N is $\geqslant 3$. We define the space of discrete solutions as

(V.17) $\quad X_\delta = S_L(\Theta, P_{N+3}(\Lambda))$.

The discrete problem is the following one : *Find* u_δ *in* X_δ *such that*

(V.18) $$\left\{ \begin{array}{ll} \Delta^2 u_\delta(x_k , \zeta_j^{\alpha+2}) = f(x_k , \zeta_j^{\alpha+2}) & , \quad -L \leqslant k \leqslant L, 1 \leqslant j \leqslant N \quad , \\ u_\delta(x_k , \pm 1) = 0 \quad , \quad -L \leqslant k \leqslant L \quad , & \\ (\partial u_\delta/\partial n)(x_k , \pm 1) = 0 \quad , \quad -L \leqslant k \leqslant L \quad . & \end{array} \right.$$

The number of equations is equal to $(2L+1)(N+4)$, which is the dimension of X_δ.

To analyse this problem, we define the discrete bilinear form on $\mathcal{C}^5(\overline{\Lambda}) \times \mathcal{C}^1(\overline{\Lambda})$ by

(V.19) $\quad b_{\alpha,N}^\ell(w,z) = \ell^4 (w,z)_{\alpha,2} - 2\ell^2 (w'',z)_{\alpha,2} + (w^{(IV)},z)_{\alpha,2}$.

We have the

<u>Proposition V.2</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *Problem* (V.18) *is equivalent to the following variational one : Find* u_δ *in* $S_L(\Theta, P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda))$ *such that, for any integer* ℓ *between* $-L$ *and* L ,

(V.20) $\quad \forall z_N \in P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \quad b_{\alpha,N}^\ell(\hat{u}_\delta^\ell , z_N) = (i_L^* f^\ell , z_N)_{\alpha,2}$.

Proof : Since the interpolation operator i_L^* is equal to the identity on $S_L(\Theta,P_{N+3}(\Lambda))$, problem (V.18) is clearly equivalent to the following one : Find u_δ in $S_L(\Theta,P_{N+3}(\Lambda))$ such that, for any integer ℓ between $-L$ and L,

(V.21) $$\left\{ \begin{array}{l} \ell^4 \hat{u}_\delta^\ell(\zeta_j^{\alpha+2}) + 2\ell^2 \hat{u}_\delta^{\ell\,''}(\zeta_j^{\alpha+2}) + \hat{u}_\delta^{\ell(IV)}(\zeta_j^{\alpha+2}) = (i_L^* f^\ell)(\zeta_j^{\alpha+2}) \quad , \quad 1 \leqslant j \leqslant N \quad , \\ \hat{u}_\delta^\ell(\pm 1) = 0 \quad , \\ (\partial \hat{u}_\delta^\ell/\partial n)(\pm 1) = 0 \quad . \end{array} \right.$$

Multiplying the first equation in (V.21) by the polynomials $(1-\zeta^2)^2 Q_j^{\alpha+2}$, $1 \leqslant j \leqslant N$, we obtain (V.20).

We must now study the properties of the form $b_{\alpha,N}^{\ell}$. We need some lemmas.

Lemma V.1 : *For any real number* $\alpha > -1$, *the following inequalities hold*

(V.22) $\quad c^{-1} \| J_{N+2}^{\alpha} \|_{0,\alpha,\Lambda}^2 \leqslant -(J_N^{\alpha+2}, J_{N+4}^{\alpha})_{\alpha,2} \leqslant c \| J_{N+2}^{\alpha} \|_{0,\alpha,\Lambda}^2$

Proof : We follow here the same lines as in the proof of Lemma IV.2 and the notation $\lambda(N) \sim \mu(N)$ again means that the quantity $\lambda(N)/\mu(N)$ is bounded by positive constants independent of N. Due to the definition (IV.1), we have

$$(J_N^{\alpha+2}, J_{N+4}^{\alpha})_{\alpha,2}$$
$$= 2\, J_N^{\alpha+2}(+1) J_{N+4}^{\alpha}(+1)\, \rho_{0,+}^{\alpha,2} + 2\,(J_N^{\alpha+2}(+1) J_{N+4}^{\alpha\,\prime}(+1) + J_N^{\alpha+2\,\prime}(+1) J_{N+4}^{\alpha}(+1)\,)\, \rho_{1,+}^{\alpha,2} \quad .$$

From (II.35), (II.36) and Lemma II.7, we deduce

$$(J_N^{\alpha+2}, J_{N+4}^{\alpha})_{\alpha,2} = 2^{2\alpha+3} \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+3)}{\alpha+3} \frac{N!}{\Gamma(N+2\alpha+5)} J_N^{\alpha+2}(+1) J_{N+4}^{\alpha}(+1)$$
$$\{ [(\alpha+2)\, N^2 + (\alpha+2)(2\alpha+5)\, N + (\alpha+3)(2\alpha+3)]$$
$$- (\alpha+1)(\alpha+3)\, [(N+4)((N+2\alpha+5)/2(\alpha+1)) + (N(N+2\alpha+5)/2(\alpha+3))] \} \quad ,$$

whence

$$(J_N^{\alpha+2}, J_{N+4}^{\alpha})_{\alpha,2} = -\, 2^{2\alpha+3}\, \Gamma(\alpha+1)\, \Gamma(\alpha+3) \frac{N!}{\Gamma(N+2\alpha+5)} J_N^{\alpha+2}(+1) J_{N+4}^{\alpha}(+1)(2N+2\alpha+7) \; .$$

Using (II.3) and the Stirling formula, we see that $-(J_N^{\alpha+2}, J_{N+4}^{\alpha})_{\alpha,2} \sim N^{-1}$. Formula (II.4), together with the Stirling's formula, also yields that $\| J_{N+2}^{\alpha} \|_{0,\alpha,\Lambda}^2 \sim N^{-1}$, whence the result.

Corollary V.1 : *For any real number* $\alpha > -1$, *the following inequalities hold*

(V.23) $\quad c^{-1} N^2 \| J_{N+2}^{\alpha} \|_{0,\alpha,\Lambda}^2 \leqslant -(J_{N+3}^{\alpha\,\prime\prime}, J_{N+3}^{\alpha})_{\alpha,2} \leqslant c\, N^2 \| J_{N+2}^{\alpha} \|_{0,\alpha,\Lambda}^2 \quad .$

Proof : Using twice formula (II.9) implies

$$(J_{N+3}^{\alpha\,\prime\prime}, J_{N+3}^{\alpha})_{\alpha,2} \sim N^2\, (J_{N+1}^{\alpha+2}, J_{N+3}^{\alpha})_{\alpha,2} \quad .$$

Next, we use the induction formula (II.6) in order to replace $J_{N+1}^{\alpha+2}$ by a combination of $\zeta J_N^{\alpha+2}$ and $J_{N-1}^{\alpha+2}$. Noting that $(J_{N-1}^{\alpha+2}, J_{N+3}^{\alpha})_{\alpha,2} = (J_{N-1}^{\alpha+2}, J_{N+3}^{\alpha})_{\alpha}$ is equal to 0, we obtain

$$(J_{N+3}^{\alpha\,\prime\prime}, J_{N+3}^{\alpha})_{\alpha,2} \sim N^2\, (\zeta J_N^{\alpha+2}, J_{N+3}^{\alpha})_{\alpha,2} \quad .$$

We replace ζJ^{α}_{N+3} by a combination of J^{α}_{N+4} and J^{α}_{N+2} , and observe that $(J^{\alpha+2}_{N}\ , J^{\alpha}_{N+2})_{\alpha,2}$ is equal to 0. That gives

$$(J^{\alpha\,''}_{N+3}\ , J^{\alpha}_{N+3})_{\alpha,2} \sim N^2\ (J^{\alpha+2}_{N}\ , J^{\alpha}_{N+4})_{\alpha,2} \quad ,$$

so that the corollary follows from Lemma V.1.

<u>Lemma V.2</u> : *For any real number* $\alpha > -1$, *the following equality holds*

$$\text{(V.24)} \qquad ((1-\zeta^2)J^{\alpha\,''}_{N}\ , (1-\zeta^2)J^{\alpha\,''}_{N})_{\alpha,0} = 2\ (\alpha+1)\ (N-1)N(2N+2\alpha+1)\ \|J^{\alpha}_{N}\|^2_{0,\alpha,\Lambda}$$

Proof : Using successively (II.8), (II.5) and (II.6), we write

$$\begin{aligned}(1-\zeta^{\alpha\,2}_{j})\ J^{\alpha\,''}_{N}(\zeta^{\alpha}_{j}) &= 2\ (\alpha+1)\ \zeta^{\alpha}_{j}\ J^{\alpha\,'}_{N}(\zeta^{\alpha}_{j}) = 2\ (\alpha+1)\ (\zeta J^{\alpha}_{N})'(\zeta^{\alpha}_{j})\\ &= 2\ ((\alpha+1)/(2N+2\alpha+1))\ [((N+1)(N+2\alpha+1)/(N+\alpha+1))\ J^{\alpha\,'}_{N+1}(\zeta^{\alpha}_{j}) + (N+\alpha)\ J^{\alpha\,'}_{N-1}(\zeta^{\alpha}_{j})]\\ &= 2\ ((\alpha+1)(N+\alpha)/(N+2\alpha))\ J^{\alpha\,'}_{N-1}(\zeta^{\alpha}_{j})\end{aligned}$$

That gives

$$((1-\zeta^2)J^{\alpha\,''}_{N}\ , (1-\zeta^2)J^{\alpha\,''}_{N})_{\alpha,0} = 4\ ((\alpha+1)(N+\alpha)/(N+2\alpha))^2\ \|J^{\alpha\,'}_{N-1}\|^2_{0,\alpha,\Lambda} \quad .$$

The quantity $\|J^{\alpha\,'}_{N+1}\|^2_{0,\alpha,\Lambda}$ can be computed from the Gauss-Lobatto quadrature formula :

$$\|J^{\alpha\,'}_{N+1}\|^2_{0,\alpha,\Lambda} = 2\ J^{\alpha\,'}_{N+1}(+1)^2\ \varrho^{\alpha,1}_{0,+} \quad ,$$

so that, due to (II.20), (II.9), (II.3) and (II.4),

$$\begin{aligned}\|J^{\alpha\,'}_{N+1}\|^2_{0,\alpha,\Lambda} &= ((N+1)(N+2\alpha+2)(2N+2\alpha+3)/2(\alpha+1))\ \ \|J^{\alpha}_{N+1}\|^2_{0,\alpha,\Lambda}\\ &= ((N+1)(N+2)(N+2\alpha+2)^2(2N+2\alpha+5)/2(\alpha+1)(N+\alpha+2)^2)\ \ \|J^{\alpha}_{N+2}\|^2_{0,\alpha,\Lambda} \quad .\end{aligned}$$

Using this formula with N+1 replaced by N-1 gives the lemma.

<u>Lemma V.3</u> : *For any real number* $\alpha > -1$, *the following equality holds for any real numbers* λ *and* μ

$$\text{(V.25)} \qquad \left|\begin{aligned}&c^{-1}\,N^3\,(\ \lambda^2\,\|J^{\alpha}_{N-1}\|^2_{0,\alpha,\Lambda} + \mu^2\,\|J^{\alpha}_{N+1}\|^2_{0,\alpha,\Lambda}\ )\\ &\qquad \leqslant ((1-\zeta^2)(\lambda\ J^{\alpha\,''}_{N-1} + \mu\ J^{\alpha\,''}_{N+1})\ , (1-\zeta^2)(\lambda\ J^{\alpha\,''}_{N-1} + \mu\ J^{\alpha\,''}_{N+1}))_{\alpha,0}\\ &\qquad\qquad \leqslant c\,N^4\,(\ \lambda^2\,\|J^{\alpha}_{N-1}\|^2_{0,\alpha,\Lambda} + \mu^2\,\|J^{\alpha}_{N+1}\|^2_{0,\alpha,\Lambda}\ ) \quad .\end{aligned}\right.$$

Proof : Thanks to (II.8), (II.6) and (II.5), we have

$$\begin{aligned}(1-\zeta^{\alpha\,2}_{j})\ J^{\alpha\,''}_{N+1}(\zeta^{\alpha}_{j}) &= 2\ (\alpha+1)\ \zeta^{\alpha}_{j}\ J^{\alpha\,'}_{N+1}(\zeta^{\alpha}_{j}) - (N+1)(N+2\alpha+2)\ J^{\alpha}_{N+1}(\zeta^{\alpha}_{j})\\ &= 2\ (\alpha+1)\ ((N+\alpha)(N+\alpha+1)/(N+2\alpha)(N+2\alpha+1))\ \zeta^{\alpha}_{j}\ J^{\alpha\,'}_{N-1}(\zeta^{\alpha}_{j})\\ &\qquad + ((N+\alpha)(N+\alpha+1)(N+2\alpha+2)/(N+2\alpha+1))\ J^{\alpha}_{N-1}(\zeta^{\alpha}_{j})\end{aligned}$$

$$= ((N+\alpha)(N+\alpha+1)/(N+2\alpha)(N+2\alpha+1))\ (1-\zeta_j^{\alpha\,2})\ J_{N-1}^{\alpha\,''}(\zeta_j^\alpha)$$
$$+ ((N+\alpha)(N+\alpha+1)(2N+2\alpha+1)/(N+2\alpha+1))\ J_{N-1}^{\alpha}(\zeta_j^\alpha)$$

That implies

$$((1-\zeta^2)(\lambda J_{N-1}^{\alpha\,''} + \mu J_{N+1}^{\alpha\,''}), (1-\zeta^2)(\lambda J_{N-1}^{\alpha\,''} + \mu J_{N+1}^{\alpha\,''}))_{\alpha,0}$$
$$= \| [\lambda + \mu ((N+\alpha)(N+\alpha+1)/(N+2\alpha)(N+2\alpha+1))]\ (1-\zeta^2) J_{N-1}^{\alpha\,''}$$
$$+ \mu ((N+\alpha)(N+\alpha+1)(2N+2\alpha+1)/(N+2\alpha+1))\ J_{N-1}^{\alpha} \|_{0,\alpha,\Lambda}^2 \quad .$$

By using twice formula (II.8), we obtain at once

$$\| (1-\zeta^2) J_{N-1}^{\alpha\,''} \|_{0,\alpha,\Lambda}^2 = \int_{-1}^{1} J_{N-1}^{\alpha}(\zeta)\ (\rho_{\alpha+2} J_{N-1}^{\alpha\,''})''(\zeta)\ d\zeta$$
$$= (N-2)(N-1)(N+2\alpha)(N+2\alpha+1)\ \| J_{N-1}^{\alpha} \|_{0,\alpha,\Lambda}^2 \quad .$$

Noting that $(1-\zeta^2) J_{N-1}^{\alpha\,''} + (N-1)(N-2)\ J_{N-1}^{\alpha}$ is of degree $\leqslant N-2$, we see that

$$\int_{-1}^{1} [(1-\zeta^2) J_{N-1}^{\alpha\,''}(\zeta) + (N-1)(N-2)\ J_{N-1}^{\alpha}(\zeta)]\ J_{N-1}^{\alpha}(\zeta)\ \rho_\alpha(\zeta)\ d\zeta = 0 \quad ,$$

hence

$$\int_{-1}^{1} (1-\zeta^2) J_{N-1}^{\alpha\,''}(\zeta)\ J_{N-1}^{\alpha}(\zeta)\ J_{N-1}^{\alpha}(\zeta)\ \rho_\alpha(\zeta)\ d\zeta = -\ (N-2)(N-1)\ \| J_{N-1}^{\alpha} \|_{0,\alpha,\Lambda}^2 \quad .$$

Finally, setting

$$\mu^* = \mu ((N+\alpha)(N+\alpha+1)/(N+2\alpha)/(N+2\alpha+1)) \quad ,$$

we have

$$((1-\zeta^2)(\lambda J_{N-1}^{\alpha\,''} + \mu J_{N+1}^{\alpha\,''}), (1-\zeta^2)(\lambda J_{N-1}^{\alpha\,''} + \mu J_{N+1}^{\alpha\,''}))_{\alpha,0} / \| J_{N-1}^{\alpha} \|_{0,\alpha,\Lambda}^2$$
$$= (\lambda + \mu^*)^2\ (N-2)(N-1)(N+2\alpha)(N+2\alpha+1)$$
$$-\ 2\ \mu^*\ (\lambda + \mu^*)\ (N-2)(N-1)(N+2\alpha)(2N+2\alpha+1) + \mu^{*2}\ (N+2\alpha)^2 (2N+2\alpha+1)^2$$
$$= (N+2\alpha)\ \{ \lambda^2\ (N-2)(N-1)(N+2\alpha+1) - 2\ \lambda\mu^*\ (N-2)(N-1)N$$
$$+ \mu^{*2}\ [N^3 + 2(7\alpha+6)N^2 + 2(\alpha+1)(10\alpha-1)N + 2(2\alpha+1)(2\alpha-1)(\alpha+1)] \} \quad .$$

Using the inequality $2\ |\lambda\mu^*| \leqslant \lambda^2 + \mu^{*2}$, we deduce that the left-hand side is less than $c\ N^4\ (\lambda^2+\mu^{*2})$, and greater than

$$(N+2\alpha)\ \{ \lambda^2\ (N-2)(N-1)(2\alpha+1)$$
$$+ \mu^{*2}\ [(14\alpha+15)N^2 + 2(10\alpha^2+9\alpha-2)N + 2(2\alpha+1)(2\alpha-1)(\alpha+1)] \} \quad .$$

It is easy to check that this last polynomial can also be written

$$(14\alpha+15)N(N-2) + 2(\alpha+1)[(10\alpha+13)N + 4\alpha^2-1]$$

hence it is larger than $N(N-2)$ for $\alpha > -1$. Noticing that both μ^{*2}/μ^2 and $\| J_{N-1}^{\alpha} \|_{0,\alpha,\Lambda}^2 / \| J_{N+1}^{\alpha} \|_{0,\alpha,\Lambda}^2$ are bounded independently of N, we obtain the two inequalities of the lemma.

The following result is proved in [BM1, Lemma 2] in the particular but not simpler case

$\alpha = 0$.

<u>Corollary V.2</u> : *For any real number* $\alpha > -1$, *the following inequalities hold for any polynomial* v *in* $\mathbb{P}_{N-1}(\Lambda)$

(V.26) $$c^{-1} N^{-1} \|(1-\zeta^2)v\|^2_{0,\alpha,\Lambda} \leqslant ((1-\zeta^2)v,(1-\zeta^2)v)_{\alpha,0} \leqslant c\,\|(1-\zeta^2)v\|^2_{0,\alpha,\Lambda} .$$

Proof : Since v belongs to $\mathbb{P}_{N-1}(\Lambda)$, we can write

$$v = \sum_{n=2}^{N+1} \tilde{v}^n J_n^{\alpha\,''} .$$

By using twice formula (II.8), we know that

$$(\rho_{\alpha+2} J_n^{\alpha\,''})'' = (n-1)n(n+2\alpha+1)(n+2\alpha+2)\,\rho_\alpha J_n^\alpha .$$

Integrating twice by parts, we obtain

(V.27) $$\|(1-\zeta^2)v\|^2_{0,\alpha,\Lambda} = \sum_{n=2}^{N+1} (\tilde{v}^n)^2 (n-1)n(n+2\alpha+1)(n+2\alpha+2)\,\|J_n^\alpha\|^2_{0,\alpha,\Lambda} .$$

On the other hand, noting that the Gauss quadrature formula is exact on $\mathbb{P}_{2N-1}(\Lambda)$, we have

(V.28) $$\left|\begin{array}{l} ((1-\zeta^2)v,(1-\zeta^2)v)_{\alpha,0} = \sum_{n=2}^{N-2} (\tilde{v}^n)^2 (n-1)n(n+2\alpha+1)(n+2\alpha+2)\,\|J_n^\alpha\|^2_{0,\alpha,\Lambda} \\ \qquad + (\tilde{v}^N)^2 ((1-\zeta^2)J_N^{\alpha\,''} , (1-\zeta^2)J_N^{\alpha\,''})_{\alpha,0} \\ \qquad + ((1-\zeta^2)(\tilde{v}^{N-1}J_{N-1}^{\alpha\,''} + \tilde{v}^{N+1}J_{N+1}^{\alpha\,''}) , (1-\zeta^2)(\tilde{v}^{N-1}J_{N-1}^{\alpha\,''} + \tilde{v}^{N+1}J_{N+1}^{\alpha\,''}))_{\alpha,0} . \end{array}\right.$$

The two last terms have been computed in Lemmas V.2 and V.3 respectively. Consequently, we obtain the desired result by comparing (V.27) and (V.28).

We are now in a position to prove the

<u>Proposition V.3</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any integer* ℓ , *the form* $b^\ell_{\alpha,N}$ *satisfies the following properties of continuity*

(V.29) $$\left|\begin{array}{l} \forall\, w_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda),\ \forall\, z_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \\ |b^\ell_{\alpha,N}(w_N , z_N)| \leqslant c\,(\ell^4 \|w_N\|^2_{0,\alpha,\Lambda} + \|w_N\|^2_{2,\alpha,\Lambda})^{1/2}\,(\ell^4 \|z_N\|^2_{0,\alpha,\Lambda} + \|z_N\|^2_{2,\alpha,\Lambda})^{1/2} , \end{array}\right.$$

and of ellipticity

(V.30) $$\left|\begin{array}{l} \forall\, w_N \in \mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda), \\ b^\ell_{\alpha,N}(w_N , w_N) \geqslant c\,(\ell^4 N^{-1} \|w_N\|^2_{0,\alpha,\Lambda} + \ell^2\, \|w_N\|^2_{1,\alpha,\Lambda} + \|w_N\|^2_{2,\alpha,\Lambda}) . \end{array}\right.$$

Proof : Let w_N and z_N be two polynomials of $\mathbb{P}_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$. We study separately each of the three terms in $b^\ell_{\alpha,N}(w_N , z_N)$.

1) We have

$$(w_N^{(IV)} , z_N)_{\alpha,2} = (w_N^{(IV)} , z_N)_\alpha ,$$

so that Proposition III.1 yields at once

(V.31) $\quad (w_N^{(IV)}, z_N)_{\alpha,2} \leq c \, \|w_N\|_{2,\alpha,\Lambda} \, \|z_N\|_{2,\alpha,\Lambda}$

and

(V.32) $\quad (w_N^{(IV)}, w_N)_{\alpha,2} \geq c \, \|w_N\|^2_{2,\alpha,\Lambda} \quad .$

2) We write the expansion

$$w_N = \sum_{n=0}^{N+3} \hat{w}^n J_n^\alpha \quad \text{and} \quad z_N = \sum_{n=0}^{N+3} \hat{z}^n J_n^\alpha \quad ;$$

since the quadrature formula is exact on $P_{2N+3}(\Lambda)$, we have

$$-(w_N'', z_N)_{\alpha,2} + (w_N'', z_N)_\alpha = -\hat{w}^{N+3} \hat{z}^{N+3} (J_{N+3}^{\alpha\,''}, J_{N+3}^\alpha)_{\alpha,2} \quad .$$

Then, Corollary V.1 yields on one hand that

(V.33) $\quad -(w_N'', w_N)_{\alpha,2} \geq -(w_N'', w_N)_\alpha \geq c \, \|w_N\|^2_{1,\alpha,\Lambda} \quad ,$

and on the other hand that

$$-(w_N'', z_N)_{\alpha,2} = -(w_N'', z_N)_\alpha + c \, \hat{w}^{N+3} \hat{z}^{N+3} N^2 \|J_{N+2}^\alpha\|^2_{0,\alpha,\Lambda} \quad .$$

Noting that, due to (II.6), the coefficient of J_{N+2}^α in the expansion of w_N' is equal to $\hat{w}^{N+3}(2N+2\alpha+5)(N+\alpha+3)/(N+2\alpha+3)$, we obtain

$$-(w_N'', z_N)_{\alpha,2} \leq c \, (\|w_N\|_{2,\alpha,\Lambda} \|z_N\|_{0,\alpha,\Lambda} + \|w_N\|_{1,\alpha,\Lambda} \|z_N\|_{1,\alpha,\Lambda}) \quad .$$

Since $H^1_\alpha(\Lambda)$ is the interpolation space of index 1/2 between $L^2_\alpha(\Lambda)$ and $H^2_\alpha(\Lambda)$, this implies

(V.34) $\quad -(w_N'', z_N)_{\alpha,2} \leq c \, (\|w_N\|_{2,\alpha,\Lambda} \|z_N\|_{0,\alpha,\Lambda} + \|w_N\|^{1/2}_{0,\alpha,\Lambda} \|w_N\|^{1/2}_{2,\alpha,\Lambda} \|z_N\|^{1/2}_{0,\alpha,\Lambda} \|z_N\|^{1/2}_{2,\alpha,\Lambda}) \quad .$

3) Finally, writing $w_N = (1-\zeta^2)^2 \tilde{w}_N$ and $z_N = (1-\zeta^2)^2 \tilde{z}_N$, where $\tilde{w}_N$ and $\tilde{z}_N$ belong to $P_{N-1}(\Lambda)$, we note that, due to (II.14),

$$(w_N, z_N)_{\alpha,2} = ((1-\zeta^2)^2 \tilde{w}_N, \tilde{z}_N)_{\alpha+2,0} \leq ((1-\zeta^2)^2 \tilde{w}_N, \tilde{w}_N)^{1/2}_{\alpha+2,0} \, ((1-\zeta^2)^2 \tilde{z}_N, \tilde{z}_N)^{1/2}_{\alpha+2,0} \quad .$$

Applying Corollary V.2 with α replaced by $\alpha+2$ and noting that $\|(1-\zeta^2)\tilde{w}_N\|_{0,\alpha+2,\Lambda}$ (resp. $\|(1-\zeta^2)\tilde{z}_N\|_{0,\alpha+2,\Lambda}$) coincide with $\|(1-\zeta^2)^2 \tilde{w}_N\|_{0,\alpha,\Lambda}$ (resp. $\|(1-\zeta^2)^2 \tilde{z}_N\|_{0,\alpha,\Lambda}$), we obtain

(V.35) $\quad (w_N, z_N)_{\alpha,2} \leq c \, \|w_N\|_{0,\alpha,\Lambda} \|z_N\|_{0,\alpha,\Lambda} \quad .$

On the other hand, Corollary V.2 implies

(V.36) $\quad (w_N, w_N)_{\alpha,2} \geq c \, N^{-1} \|w_N\|^2_{0,\alpha,\Lambda} \quad .$

Finally, the continuity property follows from (V.31), (V.34) and (V.35). The ellipticity property is a consequence of (V.32), (V.33) and (V.36).

<u>Theorem V.1</u>: *Let α satisfy $-1 < \alpha < 1$. For any f in $\mathcal{C}^0(\Omega)$, problem (V.18) has a unique solution u_δ in X_δ.*

Proof : Due to Proposition V.3, for any integer ℓ between $-L$ and L, equation (V.20) has a unique solution w_N^ℓ in $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$. The function $u_\delta = \sum_{\ell=-L}^{L} w_N^\ell \exp(i\ell x)$ is the unique solution of problem (V.18) in X_δ.

We are going to prove an error estimate between u and u_δ. If X is a Banach space, we introduce the projection operator $\pi_L^\#$ from $L^2(\Theta,X)$ onto $S_L(\Theta,X)$ which, with any function v with Fourier coefficients $\hat{v}^\ell$, $\ell \in \mathbb{Z}$, associates the trigonometric series $\pi_L^\# v = \sum_{\ell=-L}^{L} \hat{v}^\ell \exp(i\ell x)$. We recall the following result [CQ, Thm 1.1], valid for any real numbers r and s, $0 \leqslant r \leqslant s$: for any function v in $H^s_\#(\Theta,X)$,

(V.37) $\quad \|v-\pi_L^\# v\|_{H^r_\#(\Theta,X)} \leqslant c\, L^{r-s} \|v\|_{H^s_\#(\Theta,X)}$.

A similar estimate [CQ, Thm 1.2] holds for the interpolation operator if s is $> 1/2$:

(V.38) $\quad \|v-i_L^\# v\|_{H^r_\#(\Theta,X)} \leqslant c\, L^{r-s} \|v\|_{H^s_\#(\Theta,X)}$.

We conclude with the

<u>Theorem V.2</u> : *Let α satisfy $-1 < \alpha < 1$. If the solution u of problem (V.1) belongs to $H^\sigma_{\alpha,\#}(\Omega)$ for a real number $\sigma \geqslant 2$, and if the data f is such that the function $(1-y^2)^{3/2} f$ belongs to a space $H^\rho_{\alpha,\#}(\Omega)$ for a real number $\rho > 1$, the following error estimate between the solutions of problems (V.1) and (V.18) is satisfied*

(V.39) $\quad \|u-u_\delta\|_{2,\alpha,\#,\Omega} \leqslant c\, \{ (L^{2-\sigma} + N^{3-\sigma}) \|u\|_{\sigma,\alpha,\#,\Omega} + N\,(L^{-\rho} + N^{1/2-\rho}) \|(1-y^2)^{3/2} f\|_{\rho,\alpha,\#,\Omega} \}$.

Proof : We have

(V.40) $\quad \|u-u_\delta\|_{2,\alpha,\#,\Omega} \leqslant \|u-\pi_L^\# u\|_{2,\alpha,\#,\Omega} + \{ \sum_{\ell=-L}^{L} (\ell^4 \|\hat{u}^\ell-\hat{u}_\delta^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{u}^\ell-\hat{u}_\delta^\ell\|^2_{2,\alpha,\Lambda}) \}^{1/2}$.

1) Using the definition (V.5) of the spaces $H^s_{\alpha,\#}(\Omega)$, we have by (V.37)

$$\|u-\pi_L^\# u\|_{2,\alpha,\#,\Omega} \leqslant \|u-\pi_L^\# u\|_{H^2_\#(\Theta,L^2_\alpha(\Lambda))} + \|u-\pi_L^\# u\|_{L^2(\Theta,H^2_\alpha(\Lambda))}$$
$$\leqslant c\, L^{2-\sigma} (\|u\|_{H^\sigma_\#(\Theta,L^2_\alpha(\Lambda))} + \|u\|_{H^{\sigma-2}_\#(\Theta,H^2_\alpha(\Lambda))}) ,$$

whence

(V.41) $\quad \|u-\pi_L^\# u\|_{2,\alpha,\#,\Omega} \leqslant c\, L^{2-\sigma} \|u\|_{\sigma,\alpha,\#,\Omega}$.

2) For any integer ℓ, $-L \leqslant \ell \leqslant L$, Proposition V.3, together with (V.9) and (V.19), implies that, for any polynomial w_N^ℓ in $P_N(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$,

$$\ell^4 \|\hat{u}_N^\ell - w_N^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{u}_N^\ell - w_N^\ell\|^2_{2,\alpha,\Lambda} \leq c\, N\, b^\ell_{\alpha,N}(\hat{u}_N^\ell - w_N^\ell\,,\, \hat{u}_N^\ell - w_N^\ell)$$
$$\leq c\, N\, [b^\ell_\alpha(\hat{u}^\ell - w_N^\ell\,,\, \hat{u}_N^\ell - w_N^\ell) - (\hat{f}^\ell\,,\, \hat{u}_N^\ell - w_N^\ell)_\alpha + (i_L^* f^\ell\,,\, \hat{u}_N^\ell - w_N^\ell)_{\alpha,2}] \quad ,$$

(note that $b^\ell_{\alpha,N}(w_N^\ell\,;\,\hat{u}_N^\ell - w_N^\ell)$ coincides with $b^\ell_\alpha(w_N^\ell\,,\,\hat{u}_N^\ell - w_N^\ell)$), so that

$$(\,\ell^4 \|\hat{u}^\ell - \hat{u}_N^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{u}^\ell - \hat{u}_N^\ell\|^2_{2,\alpha,\Lambda}\,)^{1/2} \leq c\, N\, (\,\ell^4 \|\hat{u}^\ell - w_N^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{u}^\ell - w_N^\ell\|^2_{2,\alpha,\Lambda}\,)^{1/2}$$
$$+\, c'\, N \sup_{v_N \in P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)} \frac{(\hat{f}^\ell\,,\, v_N)_\alpha - (i_L^* f^\ell\,,\, v_N)_{\alpha,2}}{\|v_N\|_{2,\alpha,\Lambda}}\ ) \ .$$

To estimate the last term, we observe that

$$(\hat{f}^\ell\,,\, v_N)_\alpha - (i_L^* f^\ell\,,\, v_N)_{\alpha,2} = (\hat{f}^\ell - i_L^* f^\ell\,,\, v_N)_\alpha + (\hat{f}^\ell\,,\, v_N)_\alpha - (\hat{f}^\ell\,,\, v_N)_{\alpha,2}$$
$$- (\hat{f}^\ell - i_L^* f^\ell\,,\, v_N)_\alpha + (\hat{f}^\ell - i_L^* f^\ell\,,\, v_N)_{\alpha,2} \quad ,$$

so that, due to Lemma B.2,

$$(\hat{f}^\ell\,,\, v_N)_\alpha - (i_L^* f^\ell\,,\, v_N)_{\alpha,2} \leq c\,\{\, \|(1-y^2)^{3/2}\, (\hat{f}^\ell - i_L^* f^\ell)\|_{0,\alpha,\Lambda} + N^{1/2-\varrho}\, \|(1-y^2)^{3/2}\, \hat{f}^\ell\|_{\varrho,\alpha,\Lambda}$$
$$+ N^{1/2-\varrho'}\, \|(1-y^2)^{3/2}\, (\hat{f}^\ell - i_L^* f^\ell)\|_{\varrho',\alpha,\Lambda}\,\}\, \|v_N\|_{2,\alpha,\Lambda} \quad ,$$

where we have chosen $1/2 < \varrho' < \varrho - 1/2$. Choosing $w_N^\ell = \pi^{0,2}_{N,2}\hat{u}^\ell$ and applying Theorem A.6, we derive

$$\text{(V.42)} \quad \left| \begin{array}{l} (\,\ell^4 \|\hat{u}^\ell - \hat{u}_N^\ell\|^2_{0,\alpha,\Lambda} + \|\hat{u}^\ell - \hat{u}_N^\ell\|^2_{2,\alpha,\Lambda}\,)^{1/2} \\ \leq c\, N^{3-\sigma}\, (\,\ell^4 \|\hat{u}^\ell\|^2_{\sigma-2,\alpha,\Lambda} + \|\hat{u}^\ell\|^2_{\sigma,\alpha,\Lambda}\,)^{1/2} + c'\, N\, (\, \|(1-y^2)^{3/2}\, (\hat{f}^\ell - i_L^* f^\ell)\|_{0,\alpha,\Lambda} \\ \qquad + N^{1/2-\varrho}\, \|(1-y^2)^{3/2}\, \hat{f}^\ell\|_{\varrho,\alpha,\Lambda} + N^{1/2-\varrho'}\, \|(1-y^2)^{3/2}\, (\hat{f}^\ell - i_L^* f^\ell)\|_{\varrho',\alpha,\Lambda}\,) \ . \end{array} \right.$$

The final estimate follows from (V.40), (V.41), (V.42) and (V.38).

<u>Remark V.1</u> : By using the same techniques as in [BMM, § 2], one can check that, if the data f belongs to $H^\varrho_{\alpha,*}(\Omega)$ for a real number $\varrho \geq 0$, the solution u of problem (V.1) is in $H^{\varrho+4}_{\alpha,*}(\Omega)$.

<u>Remark V.2</u> : By analogy to Section IV, one could think of defining the following discrete problem : *Find* u_δ *in* X_δ *such that*

$$\text{(V.43)} \quad \left| \begin{array}{ll} \Delta^2 u_\delta(x_k\,,\,\zeta_j^{\alpha+1}) = f(x_k\,,\,\zeta_j^{\alpha+1}) \quad , & -L \leq k \leq L,\ 1 \leq j \leq N \quad ; \\ u_\delta(x_k\,,\,\pm 1) = 0 \quad , & -L \leq k \leq L \quad , \\ (\partial u_\delta/\partial n)(x_k\,,\,\pm 1) = 0 \quad , & -L \leq k \leq L \quad , \end{array} \right.$$

(where the nodes are those of the Gauss-Lobatto formula). However, it turns out that this problem is not so accurate as (V.17). Indeed, setting

$$\varphi = J^\alpha_{N+1} - ((N+2\alpha+2)(N+\alpha)(N+\alpha+1)/(N-1)N(N+2\alpha))\, J^\alpha_{N-1} \quad ,$$

we observe that $((1-\zeta^2)\varphi,\varphi)_\alpha \,/\, \|J_{N-1}^{\alpha\,\prime}\|^2_{0,\alpha+1,\Lambda}$ is bounded independently of N. On the other hand, for $1 \leqslant j \leqslant N$, $|\varphi(\zeta_j^\alpha)|$ is bounded by $(c/N)\,|J_{N-1}^{\alpha\,\prime}(\zeta_j^\alpha)|$, so that $((1-\zeta^2)\varphi,\varphi)_{\alpha,0} \,/\, \|J_{N-1}^{\alpha\,\prime}\|^2_{0,\alpha+1,\Lambda}$ is $\leqslant c\,N^{-2}$. Noting that φ vanishes in ± 1, we have found a polynomial $\psi = \varphi/(1-\zeta^2)$ in $P_{N-1}(\Lambda)$ such that

$$((1-\zeta^2)^3\psi,\psi)_{\alpha,0} \leqslant c\,N^{-2}\,((1-\zeta^2)^3\psi,\psi)_\alpha \quad .$$

Using this result with α replaced by $\alpha+1$, we obtain a polynomial w_N in $P_{N+3}(\Lambda) \cap H^2_{\alpha,0}(\Lambda)$ such that

$$(w_N\,,\,w_N)_{\alpha,1} \leqslant c\,N^{-2}\,\|w_N\|^2_{0,\alpha,\Lambda} \quad .$$

Consequently, we have proved that the constant of ellipticity of the bilinear form associated with problem (V.43) is $\leqslant c\,N^{-2}$. The convergence of the solution of this problem to the solution of problem (V.1) is not so good as the one we obtained in Theorem V.2.

Appendix A : Approximation errors.

In this first appendix we are going to derive some results concerning the weighted Sobolev spaces we use in the paper and then we shall analyse the best possible fit in these spaces. The proofs we give generalize the analysis presented in [M] that focussed on the weight $\alpha = -1/2$.

A.1. Results of interpolation between the weighted Sobolev spaces.

We begin by stating some properties of the dual spaces $H_\alpha^{-r}(\Lambda)$ of $H_{\alpha,0}^r(\Lambda)$. In the following, as is natural, we shall identify $L_\alpha^2(\Lambda)$ with its dual space. As a consequence, the differentiation in the space of distributions $\mathscr{D}'(\Lambda)$ is defined as follows :

(A.1) $\qquad \forall f \in \mathscr{D}'(\Lambda), \forall \varphi \in \mathscr{D}(\Lambda), \quad \langle df/d\zeta, \varphi\rangle = - \langle f, \rho_{-\alpha}[d(\varphi\rho_\alpha)/d\zeta] \rangle$.

Obviously the previous definition of the derivative coincides with the classical notion of differentiation for regular functions.

Let us first introduce the space $\mathscr{H}_\alpha^{-r}(\Lambda)$ of all derivatives of order r of functions of $L_\alpha^2(\Lambda)$. The following theorem gives a characterization of the dual space $H_\alpha^{-r}(\Lambda)$ of $H_{\alpha,0}^r(\Lambda)$, when $L_\alpha^2(\Lambda)$ is identified with its dual space. It is well known in the case $\alpha = 0$ and has been first established in the case $\alpha = -1/2$ in [M, Thm III.4].

<u>Theorem A.1</u> : *Let α satisfy $-1 < \alpha < 1$. For any integer r, the spaces $\mathscr{H}_\alpha^{-r}(\Lambda)$ and $H_\alpha^{-r}(\Lambda)$ coincide.*

Proof : Let us first recall that, as a consequence of [BM2, Lemma III.2], the mapping: $\varphi \to \rho_{-\alpha}[d^r(\varphi\rho_\alpha)/d\zeta^r]$ is continuous from $H_{\alpha,0}^r(\Lambda)$ into $L_\alpha^2(\Lambda)$. Then, let us consider an element φ of $H_{\alpha,0}^r(\Lambda)$; it follows that for any f in $L_\alpha^2(\Lambda)$, we have

$$\langle f, \rho_{-\alpha}[d^r(\varphi\rho_\alpha)/d\zeta^r] \rangle \leqslant c\,\|f\|_{0,\alpha,\Lambda}\,\|\varphi\|_{r,\alpha,\Lambda} \quad .$$

We derive from (A.1) that $\mathscr{H}_\alpha^{-r}(\Lambda)$ is contained in $H_\alpha^{-r}(\Lambda)$.

Conversely, the range of $H_{\alpha,0}^r(\Lambda)$ by the one-to-one mapping: $\varphi \to \rho_{-\alpha}[d^r(\varphi\rho_\alpha)/d\zeta^r]$ is a closed subspace of $L_\alpha^2(\Lambda)$. Hence, any element L of the dual space $H_\alpha^{-r}(\Lambda)$ of $H_{\alpha,0}^r(\Lambda)$ defines on that closed subspace an element $\tilde{L}$ such that

$$\forall\, \varphi \in H^r_{\alpha,0}(\Lambda), \quad \langle L, \varphi \rangle = \langle \tilde{L}, \rho_{-\alpha}[d^r(\varphi\rho_\alpha)/d\zeta^r] \rangle \quad .$$

Using now the Hahn-Banach theorem, we extend $\tilde{L}$ to an element of the dual space of the entire space $L^2_\alpha(\Lambda)$, which can be identified to a function f of $L^2_\alpha(\Lambda)$ as previously mentioned. Using once more (A.1) allows to conclude that L and $d^r f/d\zeta^r$ coincide, which implies the imbedding $H^{-r}_\alpha(\Lambda) \subset \mathcal{H}^{-r}_\alpha(\Lambda)$.

For any pair of Hilbert spaces X and Y such that X is contained in Y with a continuous imbedding and dense in Y, for any real number θ in $]0,1[$, we denote by $[X,Y]_\theta$ the interpolation space of index θ between X and Y, as defined in [LM, Chap. 1]. The following result of interpolation between the spaces $H^r_{\alpha,0}(\Lambda)$ is proved in [BM2, Lemma III.3] (similar results can be found in [Gr] in a slightly different framework).

<u>Theorem A.2</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any real numbers* p , q *and* s *which satisfy* $0 \leqslant q \leqslant s \leqslant p$ *and do not belong to* $\mathbb{N} + (1+\alpha)/2$, *the following equality holds between the spaces of interpolation*

$$[H^p_{\alpha,0}(\Lambda), H^q_{\alpha,0}(\Lambda)]_{(p-s)/(p-q)} = H^s_{\alpha,0}(\Lambda) \quad .$$

By duality, we derive the following result.

<u>Corollary A.1</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any real numbers* p , q *and* s *which satisfy* $0 \leqslant q \leqslant s \leqslant p$ *and do not belong to* $\mathbb{N} + (1+\alpha)/2$, *the following equality holds between the spaces of interpolation*

$$[H^{-q}_\alpha(\Lambda), H^{-p}_\alpha(\Lambda)]_{(s-q)/(p-q)} = H^{-s}_\alpha(\Lambda) \quad .$$

As a consequence of [LM, Chap. 1, Prop. 2.1] we obtain

<u>Lemma A.1</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any real number* s *which does not belong to* $\mathbb{N} + (1+\alpha)/2$, *the space* $[H^s_{\alpha,0}(\Lambda), H^{-s}_\alpha(\Lambda)]_{1/2}$ *is equal to* $L^2_\alpha(\Lambda)$.

From this lemma it is simple to derive as in [LM, Chap.1, Th. 6.2] the following general interpolation result.

<u>Theorem A.3</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any positive real numbers* p *and* q *which do not belong to* $\mathbb{N} + (1+\alpha)/2$ *and for any* θ *in* $]0,1[$, *set* $s = (1-\theta)p - \theta q$. *If neither* s *nor* $-s$ *belongs to* $\mathbb{N} + (1+\alpha)/2$, *the following equality holds between the spaces of interpolation*

$$[H^p_{\alpha,0}(\Lambda), H^{-q}_\alpha(\Lambda)]_\theta = \begin{cases} H^s_{\alpha,0}(\Lambda) & \text{if } s \geq 0 \ , \\ H^s_\alpha(\Lambda) & \text{if } s \leq 0 \ . \end{cases}$$

Now, we want to extend this result to the entire weighted Sobolev spaces.

Theorem A.4 : *Let* α *satisfy* $-1 < \alpha < 1$. *For any real numbers* p *and* q *which satisfy* $0 \leq q \leq p$ *and for any* θ *in* $]0,1[$, *the following equality holds between the spaces of interpolation*

$$[H^p_\alpha(\Lambda), H^q_\alpha(\Lambda)]_\theta = H^{(1-\theta)p+\theta q}_\alpha(\Lambda) \quad .$$

Proof : a) We begin with the case where q is equal to 0 and p and θp are integers. We first note that the operator $d^p/d\zeta^p$ is linear continuous from $H^p_\alpha(\Lambda)$ into $L^2_\alpha(\Lambda)$, besides, from Theorem A.1, it is also linear continuous from $L^2_\alpha(\Lambda)$ into $H^{-p}_\alpha(\Lambda)$. The principal theorem of interpolation [LM, Chap. 1, Th. 5.1] states that $d^p/d\zeta^p$ is linear continuous from $[H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_\theta$ into $[L^2_\alpha(\Lambda), H^{-p}_\alpha(\Lambda)]_\theta$. From the previous theorem, this space coincides with $H^{-\theta p}_\alpha(\Lambda)$ so that we obtain the imbedding

$$\text{(A.2)} \qquad [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_\theta \subset \{ f \in L^2_\alpha(\Lambda) \ ; \ d^pf/d\zeta^p \in H^{-\theta p}_\alpha(\Lambda) \} \quad .$$

Using now the characterization of the space $H^{-\theta p}_\alpha(\Lambda)$ given in Theorem A.1, we derive that, for any f in $L^2_\alpha(\Lambda)$ such that $d^pf/d\zeta^p$ belongs to $H^{-\theta p}_\alpha(\Lambda)$, there exists an element g in $L^2_\alpha(\Lambda)$ such that $d^pf/d\zeta^p = d^{\theta p}g/d\zeta^{\theta p}$. This means that

$$d^{\theta p}((d^{(1-\theta)p}f/d\zeta^{(1-\theta)p})-g)/d\zeta^{\theta p} = 0 \quad ,$$

hence that $(d^{(1-\theta)p}f/d\zeta^{(1-\theta)p})-g$ is a polynomial of degree $\leq \theta p-1$, thus an element of $L^2_\alpha(\Lambda)$. We conclude that $d^{(1-\theta)p}f/d\zeta^{(1-\theta)p}$ is an element of $L^2_\alpha(\Lambda)$; hence, from (A.2), we deduce

$$[H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_\theta \subset \{ f \in L^2_\alpha(\Lambda) \ ; \ d^{(1-\theta)p}f/d\zeta^{(1-\theta)p} \in L^2_\alpha(\Lambda) \} = H^{(1-\theta)p}_\alpha(\Lambda) \quad .$$

Let us recall now that the reverse imbedding was deduced in [BM2, Lemma III.3] from the results of Theorem A.2. This proves the theorem in this special case.

b) The general case is derived by using the reiteration theorem [LM, Chap. 1, Th. 6.2] in three steps. First, for any integer p and any real number θ in $]0,1[$, denoting by m the integer such that $m-1 < (1-\theta)p \leq m \leq p$, we have

$$\begin{aligned} H^{(1-\theta)p}_\alpha(\Lambda) &= [H^m_\alpha(\Lambda), H^{m-1}_\alpha(\Lambda)]_{m-(1-\theta)p} \\ &= [\ [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{1-m/p}\ , [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{1-(m-1)/p}\ ]_{m-(1-\theta)p} = [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_\theta \ . \end{aligned}$$

Next, for any positive real number p and any real number θ in $]0,1[$, denoting by s the integer

such that $s-1 < p \leqslant s$, we write

$$[H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_\theta = [\,[H^s_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{1-p/s}\,, [H^s_\alpha(\Lambda), L^2_\alpha(\Lambda)]_1\,]_\theta$$
$$= [H^s_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{1-(1-\theta)p/s} = H^{(1-\theta)p}_\alpha(\Lambda) \quad .$$

Finally, for any positive real numbers p and q, $p \leqslant q$, we derive

$$[H^p_\alpha(\Lambda), H^q_\alpha(\Lambda)]_\theta = [\,[H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_0\,, [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{1-q/p}\,]_\theta$$
$$= [H^p_\alpha(\Lambda), L^2_\alpha(\Lambda)]_{\theta(1-q/p)} = H^{(1-\theta)p+\theta q}_\alpha(\Lambda) \quad .$$

It follows from [BM2, Lemma II.5] that, for any nonnegative real number r which does not belong to $\mathbb{N} + (1+\alpha)/2$, $H^r_{\alpha,0}(\Lambda)$ is the space of all functions in $H^r_\alpha(\Lambda)$ which vanish in ± 1 together with their derivatives of order $< r - (1+\alpha)/2$. Thus, the following corollary is a simple application of [LM, Th. 13.3].

Corollary A.2 : *Let α satisfy $-1 < \alpha < 1$. For any real numbers p , q and r which satisfy $0 \leqslant r \leqslant q \leqslant p$ and such that r does not belong to $\mathbb{N} + (1+\alpha)/2$ and for any θ in $]0,1[$, the following equality holds between the spaces of interpolation*

$$[H^p_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda), H^q_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda)]_\theta = H^{(1-\theta)p+\theta q}_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda) \quad .$$

We are now in a position to study the best approximation errors in several norms.

A.2. Analysis of the best approximation in weighted norms.

We begin by analysing the best approximation of elements of $H^r_{\alpha,0}(\Lambda)$ by polynomials of $P_N(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$, when r is an integer. This is the most simple situation and can be done by induction.

Theorem A.5 : *Let α satisfy $-1 < \alpha < 1$. For any nonnegative integer r , the orthogonal projection operator $\tilde{\pi}^{0,r}_{N,r}$ from $H^r_{\alpha,0}(\Lambda)$ onto $P_N(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$ is such that , for any real number $\sigma \geqslant r$ and for any φ in $H^\sigma_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$, the following inequality holds*

$$\text{(A.3)} \qquad \|\varphi - \tilde{\pi}^{0,r}_{N,r}\varphi\|_{r,\alpha,\Lambda} \leqslant c\, N^{r-\sigma}\, \|\varphi\|_{\sigma,\alpha,\Lambda} \quad .$$

Proof : For r = 0, the result is well known and has been proven in [CQ, Thms 2.1 and 2.3] for the cases $\alpha = 0$ and $\alpha = -1/2$ and in [BM2, Thm IV.1] for the general case. The following will be an

induction process as we suppose that (A.3) holds for any integer N, with r replaced by r−1 and for any real number $\sigma \geq r-1$. Then, for any function φ in $H^\sigma_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$, we note that φ' belongs to $H^{\sigma-1}_\alpha(\Lambda) \cap H^{r-1}_{\alpha,0}(\Lambda)$. Let us define φ_N as

$$\varphi_N(\zeta) = \int_{-1}^{\zeta} [\tilde{\pi}^{0,r-1}_{N-1,r-1}(\varphi')(\xi) - m(\tilde{\pi}^{0,r-1}_{N-1,r-1}(\varphi'))(1-\xi^2)^{r-1}/m((1-\zeta^2)^{r-1})]\, d\xi \quad ,$$

where, for any ψ in $L^2_\alpha(\Lambda)$, $m(\psi)$ stands for the integral $\int_{-1}^{1} \psi(\zeta)\, d\zeta$. It is simple to note that φ_N is a polynomial of degree $\leq N$. Besides it vanishes at ± 1, as well as its first $(r-1)$ derivatives. Using now the Poincaré-Friedrichs inequality, which is valid in the weighted Sobolev spaces as is proven in [BM2, Corollary III.1] :

$$\| \varphi - \varphi_N \|_{r,\alpha,\Lambda} \leq c \| \varphi' - \varphi'_N \|_{r-1,\alpha,\Lambda} \quad ,$$

we deduce that

$$\begin{aligned} \| \varphi - \varphi_N \|_{r,\alpha,\Lambda} &\leq c \left(\| \varphi' - \tilde{\pi}^{0,r-1}_{N-1,r-1}(\varphi') \|_{r-1,\alpha,\Lambda} + | m(\tilde{\pi}^{0,r-1}_{N-1,r-1}(\varphi')) | \right) \\ &\leq c \| \varphi' - \tilde{\pi}^{0,r-1}_{N-1,r-1}(\varphi') \|_{r-1,\alpha,\Lambda} \quad . \end{aligned}$$

Using now the induction hypothesis, we deduce that

$$\| \varphi - \varphi_N \|_{r,\alpha,\Lambda} \leq c\, N^{(r-1)-(\sigma-1)} \| \varphi' \|_{\sigma-1,\alpha,\Lambda} \leq c\, N^{r-\sigma} \| \varphi \|_{\sigma,\alpha,\Lambda} \quad ,$$

which complete the induction argument since $\| \varphi - \tilde{\pi}^{0,r}_{N,r} \varphi \|_{r,\alpha,\Lambda}$ is defined as the minimum of $\| \varphi - \varphi_N \|_{r,\alpha,\Lambda}$ over all the φ_N in $P_N(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$.

Remark A.1 : It is important to note that the operator $\tilde{\pi}^{0,r}_{N,r}$ does not have optimal approximation properties in higher norms than $\| \cdot \|_{r,\alpha,\Lambda}$. For instance it is proven in [CQ, §2] and in [BM2, §IV] that there exists a function φ in $H^1_\alpha(\Lambda)$ such that the L^2_α-projection operator satisfies

$$\| \varphi - \tilde{\pi}^{0,0}_{N,0} \varphi \|_{1,\alpha,\Lambda} > c\, N^{1/2} \| \varphi \|_{1,\alpha,\Lambda} \quad ,$$

which legitimates the introduction of orthogonal projection operators in $H^r_{\alpha,0}(\Lambda)$ for any integer $r \geq 0$. One important question still remains : what are the approximation properties of the operator $\tilde{\pi}^{0,r}_{N,r}$ in lower order norms? The following theorem states that it remains optimal.

Up to now, we have not explicitly used the exact formulation of the norm on $H^r_{\alpha,0}(\Lambda)$ to define the operator $\tilde{\pi}^{0,r}_{N,r}$. In order to be able to perform a duality argument, we shall write it. As a special case of Theorem A.2, $H^r_{\alpha,0}(\Lambda)$ can be seen as the space of interpolation of index 1/2 between $L^2_\alpha(\Lambda)$ and $H^{2r}_{\alpha,0}(\Lambda)$. Consequently, there exists an unbounded self-adjoint linear operator Θ, which is positive definite and such that the domain $D(\Theta)$ of Θ in $L^2_\alpha(\Lambda)$ is $H^r_{\alpha,0}(\Lambda)$ and the domain $D(\Theta^2)$ of Θ^2 in $L^2_\alpha(\Lambda)$ is $H^{2r}_{\alpha,0}(\Lambda)$. This allows for defining a scalar product $((.,.))$

over $H^r_{\alpha,0}(\Lambda)$ as follows :

(A.4) $\quad ((u,v)) = (u,v)_\alpha + (\Theta u,\Theta v)_\alpha$.

The corresponding orthogonal projection operator will be denoted by $\pi^{0,r}_{N,r}$. The interest of this new scalar product is related to the regularity property of the following operator : let g be in $L^2_\alpha(\Lambda)$, Tg is the solution of the problem

$$\forall\, v \in H^r_{\alpha,0}(\Lambda), \quad ((Tg,v)) = (g,v)_{0,\alpha,\Lambda} \quad .$$

Then it is simple to check that T is linear and continuous from $L^2_\alpha(\Lambda)$ into $H^{2r}_{\alpha,0}(\Lambda)$. This is the main ingredient of the duality argument to derive an estimate of $\varphi-\tilde{\pi}^{0,r}_{N,r}\varphi$ in the $L^2_\alpha(\Lambda)$-norm. The reader is refered for instance to the abstract duality theorem of [M, Thm A.1] for more details. The scale of all estimates of $\varphi-\pi^{0,r}_{N,r}\varphi$ in the $H^s_\alpha(\Lambda)$-norm, $0 \leqslant s \leqslant r$, is then derived by interpolation and we obtain the following theorem.

<u>Theorem A.6</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any nonnegative integer* r , *the orthogonal projection operator* $\pi^{0,r}_{N,r}$ *from* $H^r_{\alpha,0}(\Lambda)$ *onto* $P_N(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$ *for the scalar product defined in* (A.4) *is such that , for any real numbers* σ *and* s , $0 \leqslant s \leqslant r \leqslant \sigma$, *and for any* φ *in* $H^\sigma_\alpha(\Lambda) \cap H^r_{\alpha,0}(\Lambda)$, *the following inequality holds*

(A.5) $\quad \|\varphi - \pi^{0,r}_{N,r}\varphi\|_{s,\alpha,\Lambda} \leqslant c\, N^{s-\sigma}\, \|\varphi\|_{\sigma,\alpha,\Lambda}$.

<u>Remark A.2</u> : Theorem A.6 is valid for any positive real value of r as can be derived from the same arguments as in [M]. We shall not present this extension which is not used in this paper.

The next step is the analysis of the best approximation of elements of $H^r_\alpha(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$ by polynomials of $P_N(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$, when r and p are integers, $p \leqslant r$. Let φ be an element of $H^\sigma_\alpha(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$ and φ_0 the polynomial of $P_{2r-1}(\Lambda)$ such that $\varphi - \varphi_0$ belongs to $H^r_{\alpha,0}(\Lambda)$. It is simple to note that $\pi^{0,r}_{N,r}(\varphi-\varphi_0) + \varphi_0$ is an optimal approximation of φ, more precisely,

$$\|\varphi - [\pi^{0,r}_{N,r}(\varphi-\varphi_0) + \varphi_0]\|_{s,\alpha,\Lambda} \leqslant c\, N^{s-\sigma}\, \|\varphi-\varphi_0\|_{\sigma,\alpha,\Lambda} \leqslant c'\, N^{s-\sigma}\, \|\varphi\|_{\sigma,\alpha,\Lambda} \ .$$

This leads to

<u>Theorem A.7</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any nonnegative integers* r *and* $p \leqslant r$, *there exists a projection operator* $\pi^{0,p}_{N,r}$ *from* $H^r_\alpha(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$ *onto* $P_N(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$ *such that , for any real number* $\sigma \geqslant r$ *and for any* φ *in* $H^\sigma_\alpha(\Lambda) \cap H^p_{\alpha,0}(\Lambda)$, *the following inequality holds*

(A.6) $\quad \|\varphi - \pi^{0,p}_{N,r}\varphi\|_{r,\alpha,\Lambda} \leqslant c\, N^{r-\sigma}\, \|\varphi\|_{\sigma,\alpha,\Lambda}$

<u>Remark A.3</u> : Here again, the result can be extended to any real value of r as in [M].

Finally, we can state the

<u>Corollary A.3</u> : *Let* α *satisfy* $-1<\alpha<1$. *For any nonnegative integers* r *and* p , *for any real number* $\sigma \geqslant r$ *and for any* φ *such that the function* $(1-\zeta^2)^p\,\varphi$ *belongs to* $H^\sigma_\alpha(\Lambda)$, *there exists a polynomial* φ_N *of* $P_N(\Lambda)$ *such that*

$$\text{(A.7)}\qquad \|(1-\zeta^2)^p(\varphi-\varphi_N)\|_{r,\alpha,\Lambda} \leqslant c\,N^{r-\sigma}\,\|(1-\zeta^2)^p\,\varphi\|_{\sigma,\alpha,\Lambda}\ .$$

Proof : The result is completely obvious and results from Theorem A.7 in the case where p is $\leqslant \sigma$ since φ_N can be chosen equal to $[\pi^{0,p}_{N+2p,\bar\sigma}((1-\zeta^2)^p\,\varphi)]/(1-\zeta^2)^p$, where $\bar\sigma$ denotes the integral part of σ. Otherwise, we define $(1-\zeta^2)^p\,\varphi_N$ as the orthogonal projection of φ onto $P_{N+2p}(\Lambda)\cap H^p_{\alpha,0}(\Lambda)$ for the scalar product associated with the norm : $\varphi \to \|(1-\zeta^2)^p\,\varphi\|_{r,\alpha,\Lambda}$ and we obtain the result by interpolation between the two estimates

$$\|(1-\zeta^2)^p(\varphi-\varphi_N)\|_{r,\alpha,\Lambda} \leqslant \|(1-\zeta^2)^p\,\varphi\|_{r,\alpha,\Lambda}$$

and

$$\begin{aligned}\|(1-\zeta^2)^p(\varphi-\varphi_N)\|_{r,\alpha,\Lambda} &\leqslant \|(1-\zeta^2)^p\,\varphi-[\pi^{0,p}_{N+2p,p}((1-\zeta^2)^p\,\varphi)]\|_{p,\alpha,\Lambda}\\ &\leqslant c'\,N^{r-p}\,\|(1-\zeta^2)^p\,\varphi\|_{p,\alpha,\Lambda}\ .\end{aligned}$$

Appendix B : Interpolation errors.

This appendix deals with the approximation properties of the quadrature formula analysed in the first section. Let us introduce the related operator i_{N-1}^{α} of interpolation at the internal points, defined for any function φ continuous on Λ as follows : $i_{N-1}^{\alpha}\varphi$ belongs to $P_{N-1}(\Lambda)$ and satisfies

(B.1) $\qquad (i_{N-1}^{\alpha}\,\varphi)(\zeta_j^{\alpha}) = \varphi(\zeta_j^{\alpha}) \quad , 1 \leqslant j \leqslant N \quad .$

The following estimate is well-known in the case $\alpha = 0$ [CQ, Thm 3.2], it can easily be proved for any $\alpha > -1$ in a similar way : for any real number $\sigma > \sup\{1/2, (1+\alpha)/2\}$ and for any function φ in $H_{\alpha}^{\sigma}(\Lambda)$

(B.2) $\qquad \|\varphi - i_{N-1}^{\alpha}\varphi\|_{0,\alpha,\Lambda} \leqslant c\, N^{1/2-\sigma}\, \|\varphi\|_{\sigma,\alpha,\Lambda} \quad .$

The previous inequality implies as a special case that, for any integer $m \geqslant 1$ and for any smooth function φ, the quantity $\|(1-\zeta^2)^{m/2}\,(\varphi - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda}$ tends to 0. The following lemma gives a precise form of this result.

<u>Lemma B.1</u> : *Let* α *satisfy* $-1 < \alpha < 1$. *For any integer* $m \geqslant 1$, *define* k *as being equal to* $m/2$ *or* $(m+1)/2$. *For any real number* $\sigma > \sup\{1/2, (1+\alpha)/2\}$ *and for any* φ *such that the function* $(1-\zeta^2)^k\,\varphi$ *belongs to* $H_{\alpha}^{\sigma}(\Lambda)$, *the following inequality holds*

(B.3) $\qquad \|\,(1-\zeta^2)^k\,(\varphi - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda} \leqslant c\, N^{1/2-\sigma}\, \|\,(1-\zeta^2)^k\,\varphi\,\|_{\sigma,\alpha,\Lambda} \quad .$

Proof : Let φ_N be any element of $P_{N-1}(\Lambda)$. We use the triangular inequality

(B.4) $\qquad \|(1-\zeta^2)^k\,(\varphi - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda} \leqslant \|(1-\zeta^2)^k\,(\varphi-\varphi_N)\,\|_{0,\alpha,\Lambda} + \|\,(1-\zeta^2)^k\,(\varphi_N - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda}\,.$

Let us recall that the quadrature formula $(.,.)_{\alpha,m}$ is exact on $\mathbb{P}_{2N+2m-1}(\Lambda)$. In the case $2k = m$, we deduce that

$$\|(1-\zeta^2)^k\,(\varphi_N - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda}^2 = \sum_{j=1}^{N} (\varphi-\varphi_N)^2(\zeta_j^{\alpha+m})(1-(\zeta_j^{\alpha+m})^2)^{2k}\,\rho_j^{\alpha,m} \quad .$$

In the case $2k = m+1$, we have

$$\|(1-\zeta^2)^k\,(\varphi_N - i_{N-1}^{\alpha+m}\varphi)\,\|_{0,\alpha,\Lambda}^2 = \|(1-\zeta^2)^{1/2}\,(\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha+m,\Lambda}^2 \quad .$$

Using exactly the same techniques as for [BM1, Lemma 1], we prove that, for any $\alpha > -1$ and for any $\psi_N = \sum_{n=1}^{N} \alpha_n\, J_n^{\alpha\prime}$ in $P_{N-1}(\Lambda)$,

$$\sum_{j=1}^{N} (1-(\zeta_j^\alpha)^2)\, \psi_N{}^2(\zeta_j^\alpha)\, \varrho_j^{\alpha,0}$$
$$= \sum_{n=1}^{N-1} \alpha_n^2\, n(n+2\alpha+1)\, \|J_n^\alpha\|_{0,\alpha,\Lambda}^2 + \alpha_N^2 \sum_{j=1}^{N} (1-(\zeta_j^\alpha)^2)\, J_N^{\alpha,2}(\zeta_j^\alpha)\, \varrho_j^{\alpha,0}$$
$$\square \sum_{n=1}^{N-1} \alpha_n^2\, n(n+2\alpha+1)\, \|J_n^\alpha\|_{0,\alpha,\Lambda}^2 + \alpha_N^2\, N(2N+2\alpha+1)$$
$$\geqslant \sum_{n=1}^{N} \alpha_n^2\, n(n+2\alpha+1)\, \|J_n^\alpha\|_{0,\alpha,\Lambda}^2 = \|(1-\zeta^2)^{1/2}\, \psi_N\|_{0,\alpha,\Lambda}^2 \ .$$

This estimate, with α replaced by $\alpha+m$ and $\psi_N = \varphi_N - i_{N-1}^{\alpha+m}\varphi$, gives

$$\|(1-\zeta^2)^k\, (\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha,\Lambda}^2 \leqslant \sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)\, (\varphi-\varphi_N)^2(\zeta_j^{\alpha+m})\, \varrho_j^{\alpha+m,0}$$
$$\leqslant \sum_{j=1}^{N} (\varphi-\varphi_N)^2(\zeta_j^{\alpha+m})(1-(\zeta_j^{\alpha+m})^2)^{2k}\, \varrho_j^{\alpha,m} \ .$$

Consequently, in both cases, we obtain

$$\|(1-\zeta^2)^k\, (\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha,\Lambda}^2 \leqslant \|(1-\zeta^2)^{k+\alpha/2}\, (\varphi-\varphi_N)\|_{L^\infty(\Lambda)}^2 \sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^{-\alpha}\, \varrho_j^{\alpha,m}$$

If α is $\leqslant 0$, it is an easy matter to prove that the sum on the right-hand side is bounded : indeed, all the $(1-(\zeta_j^{\alpha+m})^2)^{-\alpha}$ are $\leqslant 1$ and all the corresponding weights are positive; since the constant polynomial is exactly integrated by the formula, we derive

$$\sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^{-\alpha}\, \varrho_j^{\alpha,m} \leqslant \sum_{j=1}^{N} \varrho_j^{\alpha,m} \leqslant 2.$$

If α satisfies $0 \leqslant \alpha \leqslant 1$, we recall that, due to (II.14), we can write

$$\sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^{-\alpha}\, \varrho_j^{\alpha,m} = \sum_{j=1}^{N} (1-(\zeta_j^{\alpha+m})^2)^{1-\alpha}\, \varrho_j^{\alpha-1,m+1}$$

and the same argument as before can be used since now $1-\alpha$ is $\geqslant 0$. In conclusion, we can state that there exists a constant C such that

$$\|(1-\zeta^2)^k\, (\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha,\Lambda}^2 \leqslant C\, \|(1-\zeta^2)^{k+\alpha/2}\, (\varphi-\varphi_N)\|_{L^\infty(\Lambda)}^2 \ .$$

The Gagliardo-Nirenberg inequality now yields to the bound

$$\|(1-\zeta^2)^k\, (\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha,\Lambda}^2 \leqslant c\, \|(1-\zeta^2)^{k+\alpha/2}\, (\varphi-\varphi_N)\|_{L^2(\Lambda)}\, \|(1-\zeta^2)^{k+\alpha/2}\, (\varphi-\varphi_N)\|_{H^1(\Lambda)} \ .$$

Besides, since the multiplication by $\varrho_{-\alpha/2}$ is an isomorphism from $L^2(\Lambda)$ onto $L^2_\alpha(\Lambda)$ and from $H^1_0(\Lambda)$ onto $H^1_{\alpha,0}(\Lambda)$ [BM2, Lemma III.2], we derive that

$$\text{(B.5)} \qquad \|(1-\zeta^2)^k\, (\varphi_N - i_{N-1}^{\alpha+m}\varphi)\|_{0,\alpha,\Lambda}^2 \leqslant c\, \|(1-\zeta^2)^k\, (\varphi-\varphi_N)\|_{0,\alpha,\Lambda}\, \|(1-\zeta^2)^k\, (\varphi-\varphi_N)\|_{1,\alpha,\Lambda} \ .$$

The inequality (B.3) is now a simple consequence of (B.4), (B.5) and Corollary A.1.

We can state the properties of the quadrature formula.

<u>Lemma B.2</u> : *Let α satisfy $-1 < \alpha < 1$. For any integer $m \geqslant 1$ and for any real number $\sigma > \sup\{1/2, (1+\alpha)/2\}$, for any f such that the function $(1-\zeta^2)^{(m+1)/2} f$ belongs to $H^\sigma_\alpha(\Lambda)$, the following inequality holds*

$$\text{(B.6)} \qquad \sup_{v_N \in P_{N+2m-1}(\Lambda) \cap H^m_{\alpha,0}(\Lambda)} \frac{(f,v_N)_\alpha - (f,v_N)_{\alpha,m}}{\|v_N\|_{m,\alpha,\Lambda}} \leqslant c\, N^{1/2-\sigma}\, \|(1-\zeta^2)^{(m+1)/2} f\|_{\sigma,\alpha,\Lambda} \ .$$

Proof : Since the quadrature formula is exact on $P_{2N+2m-1}(\Lambda)$, we have for any v_N in $P_{N+2m-1}(\Lambda) \cap H^m_{\alpha,0}(\Lambda)$ and any f_N in $P_{N-1}(\Lambda)$

$$(f,v_N)_\alpha - (f,v_N)_{\alpha,m} = (f-f_N,v_N)_\alpha - (i^{\alpha+m}_{N-1}f-f_N,v_N)_\alpha \quad .$$

Next, we note that v_N can be written $(1-\zeta^2)^m \tilde{v}_N$, where $\tilde{v}_N$ belongs to $P_{N-1}(\Lambda)$. Moreover, using once more the fact that the multiplication by $\rho_{\alpha/2}$ is an isomorphism from $H^k_{\alpha,0}(\Lambda)$ onto $H^k_0(\Lambda)$ for any $k \geq 0$ and that the multiplication by $(1-\zeta^2)^{-m}$ is continuous from $H^m_0(\Lambda)$ into $L^2(\Lambda)$ [LM, Th. 11.2 and 11.3], we have

$$\|\tilde{v}_N\|_{0,\alpha,\Lambda} = \|\tilde{v}_N\,\rho_{\alpha/2}\|_{0,\Lambda} \leq c\,\|(1-\zeta^2)^m\,\tilde{v}_N\,\rho_{\alpha/2}\|_{m,\Lambda} = c\,\|v_N\,\rho_{\alpha/2}\|_{m,\Lambda} \leq c'\,\|v_N\|_{m,\alpha,\Lambda} \quad .$$

Thus, we derive

$$\begin{aligned}(f,v_N)_\alpha - (f,v_N)_{\alpha,m} &= (f-f_N,\tilde{v}_N)_{\alpha+m} - (i^{\alpha+m}_{N-1}f-f_N,\tilde{v}_N)_{\alpha+m} \\ &\leq c\,[\|(1-\zeta^2)^m\,(f-f_N)\|_{0,\alpha,\Lambda} + \|(1-\zeta^2)^m\,(i^{\alpha+m}_{N-1}f-f_N)\|_{0,\alpha,\Lambda}]\,\|\tilde{v}_N\|_{0,\alpha,\Lambda} \\ &\leq c\,[\|(1-\zeta^2)^m\,(f-f_N)\|_{0,\alpha,\Lambda} + \|(1-\zeta^2)^m\,(i^{\alpha+m}_{N-1}f-f_N)\|_{0,\alpha,\Lambda}]\,\|v_N\|_{m,\alpha,\Lambda} \quad .\end{aligned}$$

Since $(1-\zeta^2)^m$ is upbounded by $(1-\zeta^2)^{(m+1)/2}$, we obtain

$$\begin{aligned}&(f,v_N)_\alpha - (f,v_N)_{\alpha,m} \\ &\quad\leq c\,[\|(1-\zeta^2)^{(m+1)/2}\,(f-f_N)\|_{0,\alpha,\Lambda} + \|(1-\zeta^2)^{(m+1)/2}\,(i^{\alpha+m}_{N-1}f-f_N)\|_{0,\alpha,\Lambda}]\,\|v_N\|_{m,\alpha,\Lambda} \quad .\end{aligned}$$

and the result follows from Lemma B.1 together with Corollary A.1.

<u>Remark B.1</u> : We actually proved the stronger result

$$\sup_{v_N \in P_{N+2m-1}(\Lambda) \cap H^\ell_{\alpha,0}(\Lambda)} \frac{(f,v_N)_\alpha - (f,v_N)_{\alpha,m}}{\|v_N\|_{\ell,\alpha,\Lambda}} \leq c\,N^{1/2-\sigma}\,\|(1-\zeta^2)^{(m+1)/2}\,f\|_{\sigma,\alpha,\Lambda} \quad ,$$

where ℓ is equal to $(m+1)/2$ if m is odd and is $> (m+1)/2$ if m is even. However, we do not need this improvement.

<u>Remark B.2</u> : In the previous analysis, we only need the Lagrange interpolation operator. With the generalized Gauss type formula, it seems natural to associate the Hermite interpolation operator $\mathcal{J}^\alpha_{N+2m-1}$: for any function φ in $\mathcal{C}^{m-1}(\overline{\Lambda})$, $\mathcal{J}^\alpha_{N+2m-1}\varphi$ belongs to $P_{N+2m-1}(\Lambda)$ and satisfies

$$\text{(B.6)} \qquad \left\{\begin{array}{ll}(\mathcal{J}^\alpha_{N+2m-1}\varphi)(\zeta^{\alpha+m}_i) = \varphi(\zeta^{\alpha+m}_i) & ,\ 1 \leq i \leq m \quad , \\ (d^k/d\zeta^k)(\mathcal{J}^\alpha_{N+2m-1}\varphi)(\pm 1) = (d^k\varphi/d\zeta^k)(\pm 1) & ,\ 0 \leq k \leq m-1 \quad .\end{array}\right.$$

However, the approximation properties of this operator seems poor when m is larger than 1. Indeed, in the case m = 1, the following estimate [BM2, Lemma V.9] holds for any function φ in

$H^\sigma_\alpha(\Lambda)$, $\sigma > \sup\{1/2, (1+\alpha)/2\}$,

(B.7) $$\|\varphi-(\mathcal{J}^\alpha_{N+1}\varphi)\|_{0,\alpha,\Lambda} \leqslant c\, N^{1/2-\sigma}\,\|\varphi\|_{\sigma,\alpha,\Lambda}$$

But in the case $m = 2$, due to Corollary V.2, the only interpolation result we are able to prove by using the same techniques is that, for any function φ in $H^\sigma_\alpha(\Lambda)$, $\sigma > \sup\{3/2, (3+\alpha)/2\}$,

(B.8) $$\|\varphi-(\mathcal{J}^\alpha_{N+3}\varphi)\|_{0,\alpha,\Lambda} \leqslant c\, N^{3/2-\sigma}\,\|\varphi\|_{\sigma,\alpha,\Lambda}$$

Appendix C : Nodes and weights.

We end this paper by giving numerical examples of the nodes and weights involved in the generalized quadrature formula (II.10). Let us briefly recall how they are computed (we refer to [GK][KE] for more general techniques). First, setting for a while $J_n^{\alpha *} = J_n^\alpha / \|J_n^\alpha\|_{0,\alpha,\Lambda}$, $n \in \mathbb{N}$, we have by (II.4) and (II.5)

$$(C.1)\qquad \zeta \begin{bmatrix} J_0^{\alpha *} \\ J_1^{\alpha *} \\ \cdots \\ \cdots \\ J_{N-1}^{\alpha *} \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & 0 & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \beta_{N-1} & 0 \end{bmatrix} \begin{bmatrix} J_0^{\alpha *} \\ J_1^{\alpha *} \\ \cdots \\ \cdots \\ J_{N-1}^{\alpha *} \end{bmatrix} + \beta_N \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ J_N^{\alpha *} \end{bmatrix},$$

with

$$(C.2)\qquad \beta_1 = \frac{1}{\sqrt{2\alpha+3}} \quad \text{and} \quad \beta_n = \sqrt{\frac{n\,(n+2\alpha)}{4\,(n+\alpha)^2 - 1}}\;, n \geq 2\;.$$

Hence, the nodes ζ_j^α, $1 \leq j \leq N$, of the Gauss formula are simply the eigenvalues of the previous symmetric tridiagonal matrix; due to formula (II.12), the weights $\rho_j^{\alpha,0}$, $1 \leq j \leq N$, are equal to $\|J_0^\alpha\|^2_{0,\alpha,\Lambda}\, X_{j1}{}^2$, where X_{j1} is the first component of one of the corresponding eigenvectors with euclidean norm equal to 1. Therefore, nodes and weights can be computed for instance by a standard QR-algorithm (see [WR]).

The internal nodes $\zeta_j^{\alpha+m}$ and weights $\rho_j^{\alpha,m}$, $1 \leq j \leq N$, of the generalized quadrature formula (II.10) are obtained by using exactly the same algorithm with α replaced by $\alpha+m$, then applying formula (II.14). As far as the boundary weights are concerned, the first pair $\rho_{0,\pm}^{\alpha,1}$ is calculated from formula (II.20). Then, for $m \geq 2$, the other pairs $\rho_{k,\pm}^{\alpha,m}$, $1 \leq k \leq m-1$, are obtained by induction on m, from the triangular linear system (II.21)(II.22); finally, the pair $\rho_{0,\pm}^{\alpha,m}$ is computed from formula (II.29) or (II.31).

Figures 1, 2 and 3 represent the nonnegative zeros of the polynomial J_N^α respectively for N = 6, 9 and 12, as a function of α, $-1 < \alpha \leq 3$ (recall that we are mainly interested with values of α between -1 and 1, and values of m equal to 0, 1 or 2, so that the range of $\alpha+m$ is $]-1,3[$). It can be observed that, when α grows from -1 to 3, these zeros decrease slowly (but not linearly).

Figures 4, 5 and 6 represent all the zeros of the same polynomials for several values of α, namely for $\alpha = -3/4, -1/2, -1/4, \ldots, 11/4, 3$. As is well-known, for fixed values of α and N, these zeros are not evenly distributed, but they cluster in the neighbourhood of ± 1.

Finally, Tables I to X can be used to compute $\int_{-1}^{1} \Phi(\zeta)\, \rho_\alpha(\zeta)\, d\zeta$, for $\alpha = -1/2, -1/4, 0, 1/4$ and $1/2$, by a quadrature formula involving respectively 12 or 40 internal nodes; each table gives the nodes and weights of formula (II.10) for $m = 0, 1$ and 2. Note that, as foreseen, the nodes of the formula for $\alpha = -1/2$ and $m = 1$ or 2 coincide respectively with those of the formula for $\alpha = 1/2$ and $m = 0$ or 1. As it is well-known, in the Chebyshev case $\alpha = -1/2$, for $m = 0$, the internal weights are all equal to the same quotient of π; the same is true in the case $m = 1$ and moreover the boundary weights are the half of the internal weights; these properties are no longer valid when m is equal to 2.

<u>Example C.1</u> : Let us approximate the quantity $\mathfrak{J} = (\pi/4) \int_{-1}^{1} \cos(\pi\zeta/2)\, d\zeta = 1$ by using the quadrature formula (II.10) with $N = 6$ internal nodes, successively for $m = 0, 1$ and 2. We obtain

$m = 0 : \mathfrak{J} \simeq 0.9999999997386354$

$m = 1 : \mathfrak{J} \simeq 0.9999999999989776$

$m = 2 : \mathfrak{J} \simeq 0.9999999999999963$.

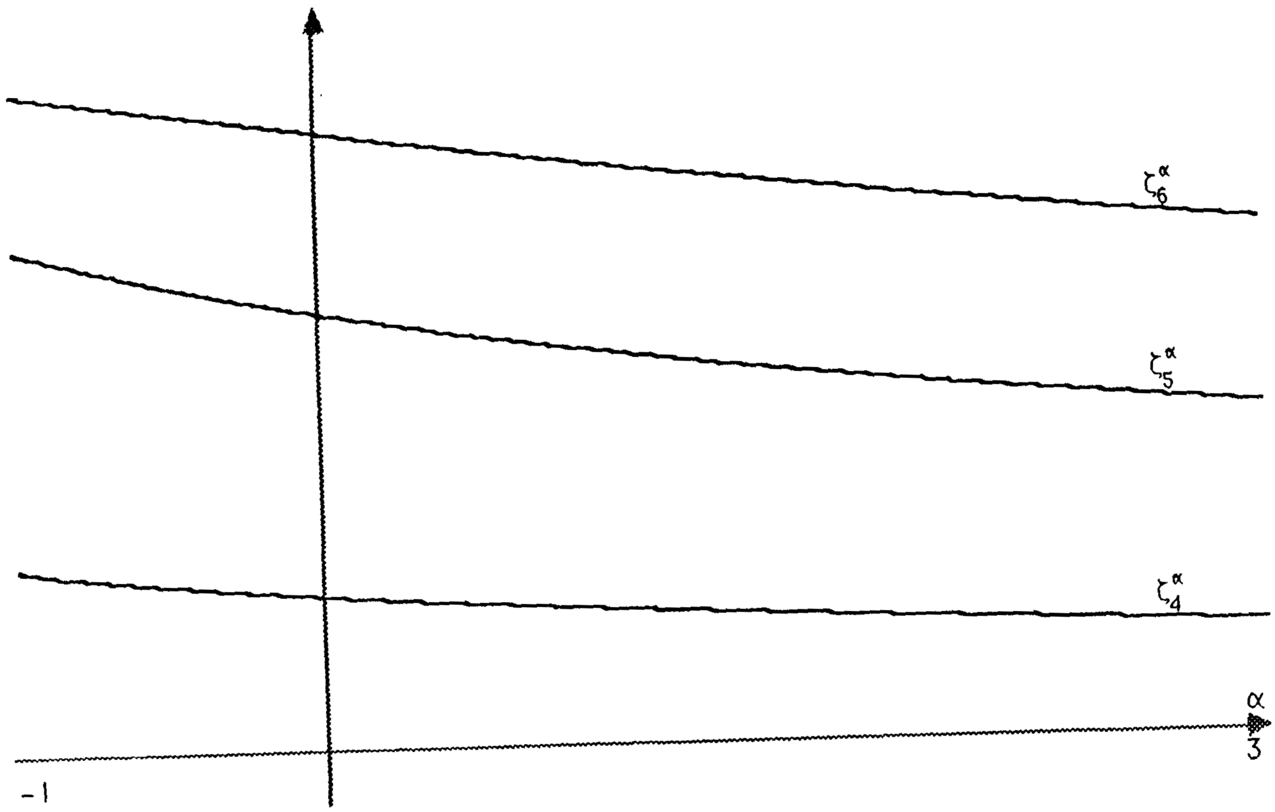

Figure 1

The three nonnegative zeros of J_6^α as a function of α.

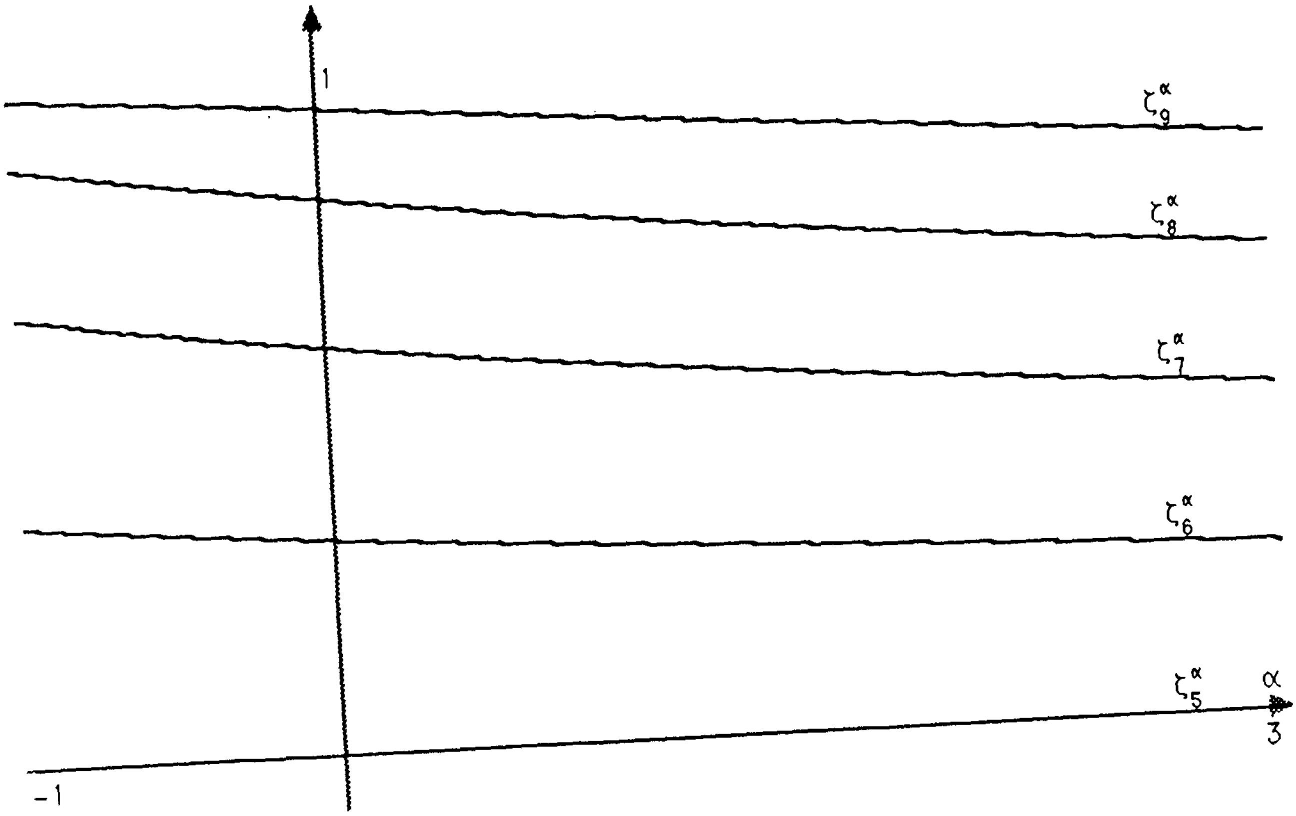

Figure 2

The five nonnegative zeros of J_9^α as a function of α.

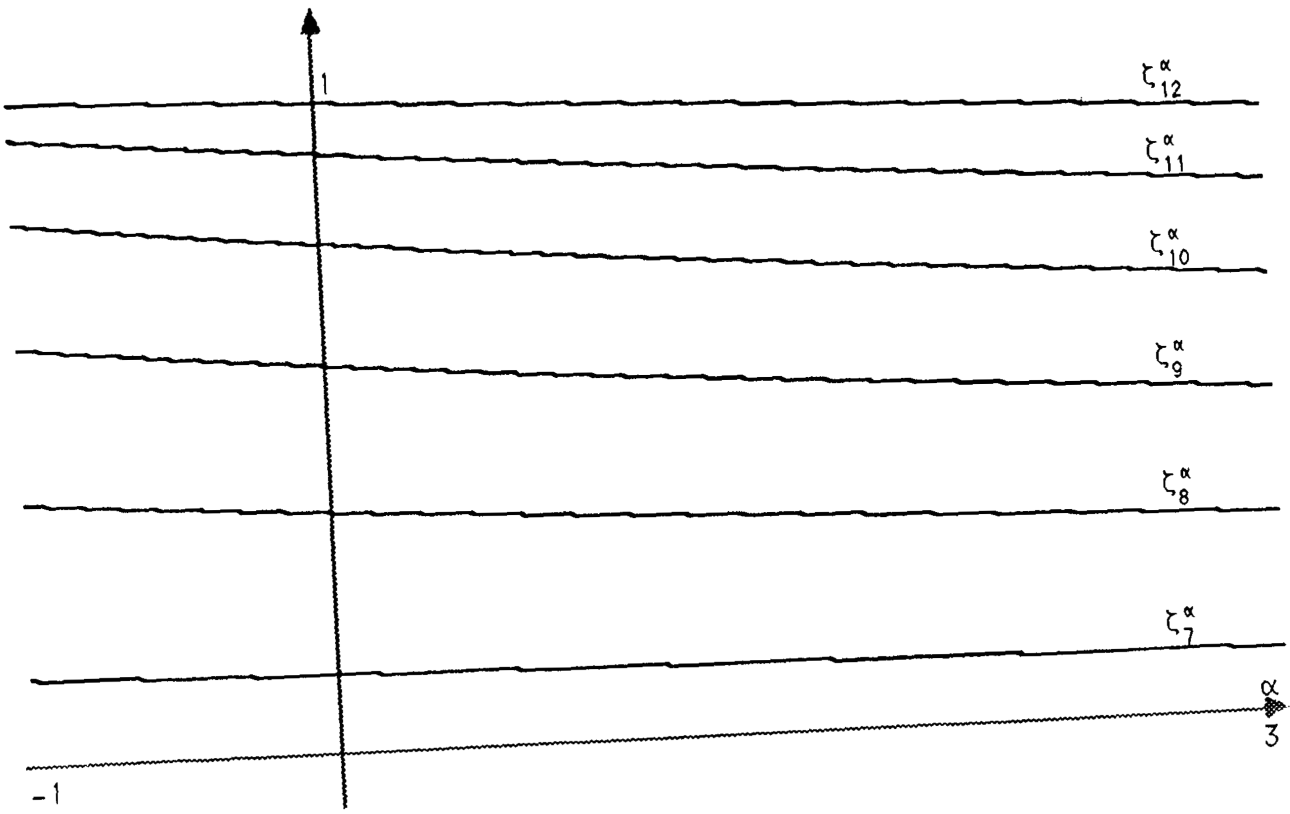

Figure 3

The six nonnegative zeros of J_{12}^{α} as a function of α.

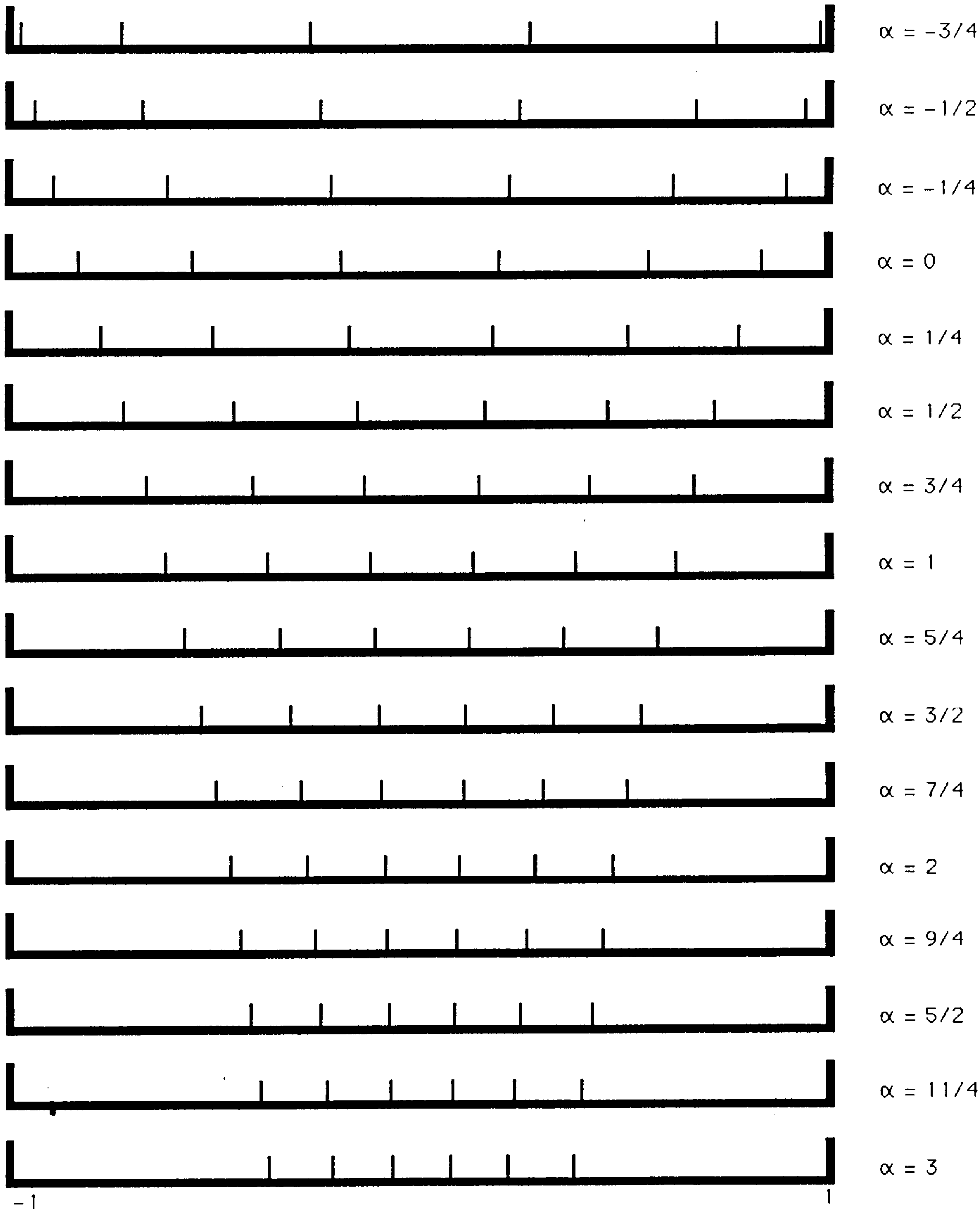

Figure 4

The zeros of J_6^α, $-1 < \alpha \leqslant 3$, $\alpha \in \mathbb{N}/4$.

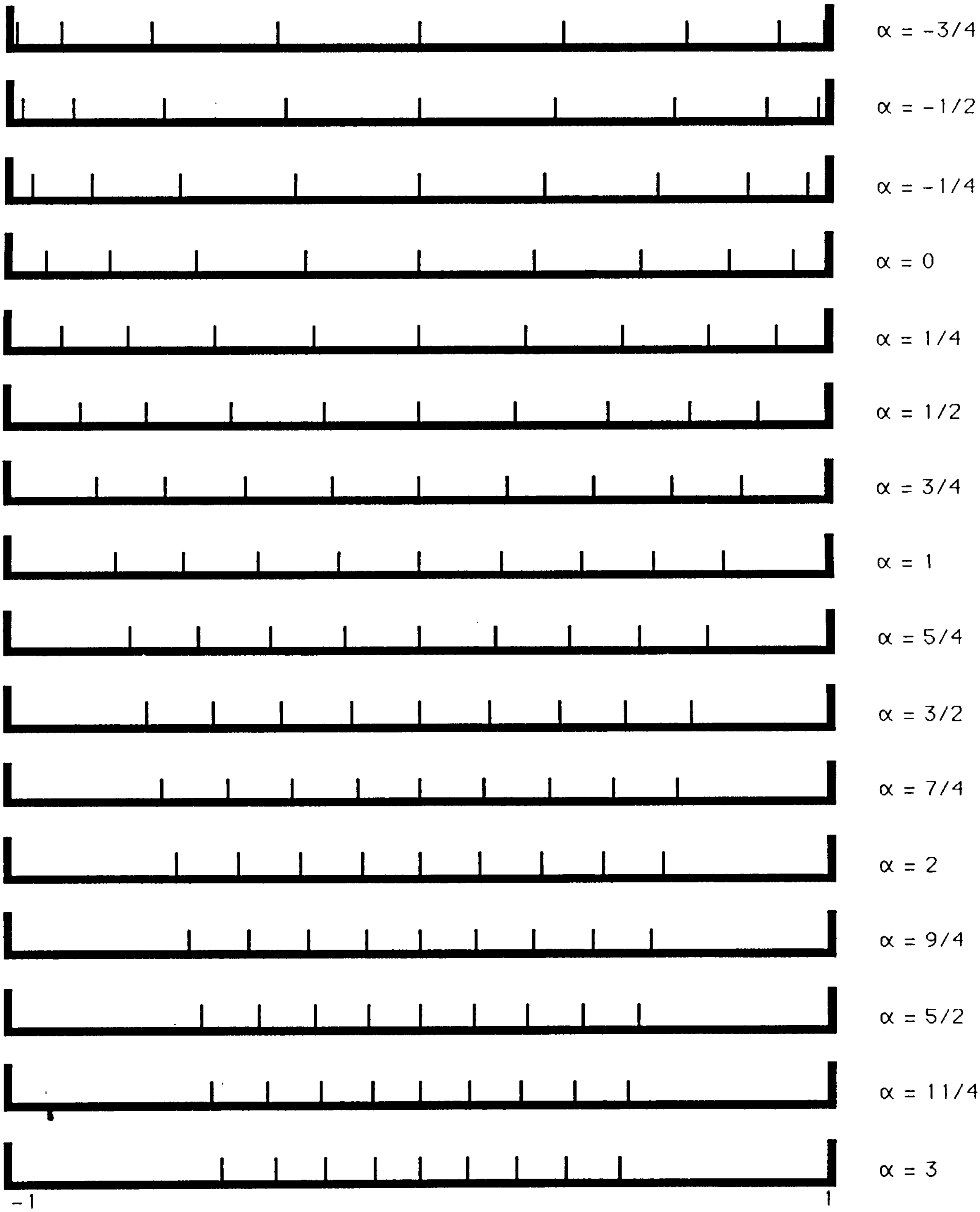

Figure 5

The zeros of J_9^α, $-1 < \alpha \leqslant 3$, $\alpha \in \mathbb{N}/4$.

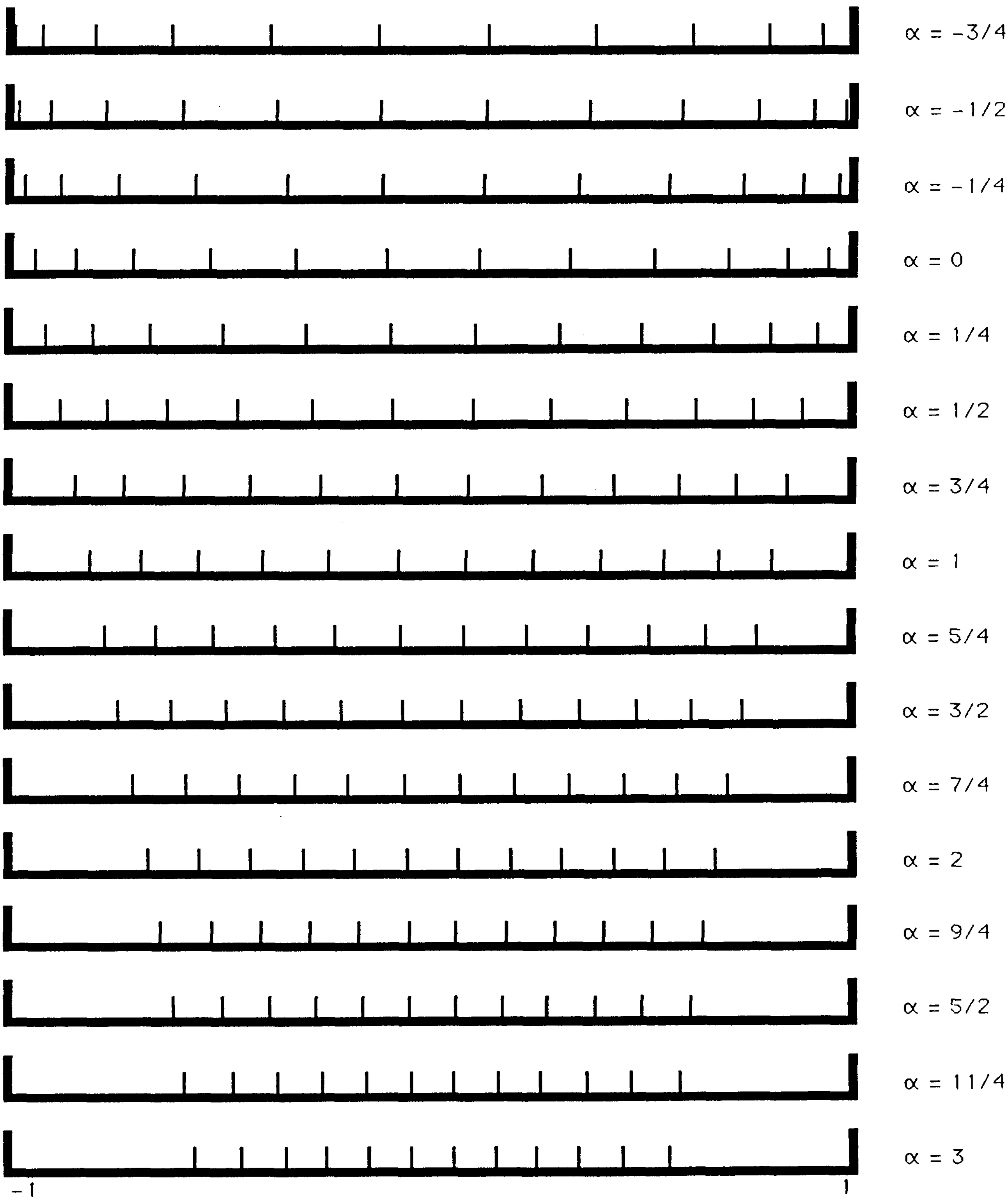

Figure 6

The zeros of J_{12}^{α}, $-1 < \alpha \leqslant 3$, $\alpha \in \mathbb{N}/4$.

Table I

The nodes and weights of three quadrature formulas with $N = 12$ internal nodes for $\alpha = -1/2$.

Gauss formula

Nodes	Weights
± 0.1305261922200516	0.2617993877991494
± 0.3826834323650898	0.2617993877991494
± 0.6087614290087206	0.2617993877991494
± 0.7933533402912352	0.2617993877991494
± 0.9238795325112868	0.2617993877991494
± 0.9914448613738104	0.2617993877991494

Gauss-Lobatto formula

Nodes	Weights
± 0.1205366802553231	0.2416609733530610
± 0.3546048870425356	0.2416609733530610
± 0.5680647467311558	0.2416609733530610
± 0.7485107481711011	0.2416609733530610
± 0.8854560256532099	0.2416609733530610
± 0.9709418174260520	0.2416609733530610
± 1	0.1208304866765305

Generalized Gauss formula ($m = 2$)

Nodes	Weights
± 0.1125386591664354	0.2255650031484414
± 0.3319729790280478	0.2256927708010325
± 0.5347613738021686	0.2260111247664517
± 0.7107361043825368	0.2267249411475805
± 0.8510759734996212	0.2285739151482831
± 0.9487595481458028	0.2359238140904020
± 1	$k = 0$: 0.2023047576927054
	$k = 1$: $-(\pm)$ 0.0008630749048324

Table II

The nodes and weights of three quadrature formulas with N = 12 internal nodes for $\alpha = -1/4$.

Gauss formula

Nodes	Weights
± 0.1277976132681700	0.2552414438389092
± 0.3750336874849222	0.2467415580312443
± 0.5977391874608044	0.2294029769975812
± 0.7813478001040569	0.2022815344570018
± 0.9138534350790553	0.1628305472653609
± 0.9866236439839729	0.1016421741454949

Gauss-Lobatto formula

Nodes	Weights
± 0.1183782733100242	0.2364792066164156
± 0.3485109774639786	0.2297804338676144
± 0.5591427246672460	0.2161669597909127
± 0.7384872708173654	0.1950703244754963
± 0.8765077811134355	0.1651230971924475
± 0.9654710290607673	0.1224822478290640
± 1	0.0330379649636415

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.1107761543915436	0.2213364789959417
± 0.3269666173060262	0.2159939810406281
± 0.5273314294625529	0.2051830215882793
± 0.7021743770629110	0.1885960741990150
± 0.8430406218744945	0.1656126241288492
± 0.9431557294997915	0.1353053751726552
± 1	k = 0 : 0.0661126796102237
	k = 1 : − (±) 0.0003858716708768

Table III

The nodes and weights of three quadrature formulas with $N = 12$ internal nodes for $\alpha = 0$.

Gauss formula

Nodes	Weights
± 0.1252334085114689	0.2491470458134028
± 0.3678314989981802	0.2334925365383548
± 0.5873179542866174	0.2031674267230659
± 0.7699026741943047	0.1600783285433462
± 0.9041172563704749	0.1069393259953184
± 0.9815606342467193	0.0471753363865118

Gauss–Lobatto formula

Nodes	Weights
± 0.1163318688837039	0.2316127944684571
± 0.3427240133427128	0.2191262530097708
± 0.5506394029286471	0.1948261493734161
± 0.7288685990913261	0.1600218517629521
± 0.8678010538303473	0.1165866558987117
± 0.9599350452672609	0.0668372844976813
± 1	0.0109890109890110

Generalized Gauss formula ($m = 2$)

Nodes	Weights
± 0.1090939783245157	0.217333937524949556
± 0.3221818845733797	0.207173867770578608
± 0.5202087732305217	0.187330255034571925
± 0.6939208718647531	0.158736180770208859
± 0.8352126561026332	0.122750599140707938
± 0.9375597349081076	0.081217284301107650
± 1	$k = 0$: 0.0254578754578755 $k = 1$: −(±) 0.0001831501831502

Table IV

The nodes and weights of three quadrature formulas with $N = 12$ internal nodes for $\alpha = 1/4$.

Gauss formula

Nodes	Weights
± 0.1228177004683573	0.2434645201768296
± 0.3610342534563323	0.2217497363900194
± 0.5774424805951338	0.1816128528241748
± 0.7589706245718064	0.1292869893886836
± 0.8946555913178371	0.0733734189289260
± 0.9763175017102367	0.0245316670554065

Gauss-Lobatto formula

Nodes	Weights
± 0.1143880994004779	0.2270313048465768
± 0.3372188010378834	0.2095212174757884
± 0.5425220077978783	0.1767797234841329
± 0.7196266099840568	0.1331496497891043
± 0.8593266815884500	0.0846685013893612
± 0.9543580065299194	0.0387135239093696
± 1	0.0041552638697068

Generalized Gauss formula ($\dot{m} = 2$)

Nodes	Weights
± 0.1074862066759131	0.2135381488877235
± 0.3176027699572138	0.1991205440899331
± 0.5133723208350232	0.1719112391486291
± 0.6859569006710141	0.1350266051900061
± 0.8275844178690692	0.0928336671288014
± 0.9319824128236114	0.0507035575084562
± 1	$k = 0$: 0.0108854228104905
	$k = 1$: −(±) 0.0000913914340845

Table V

The nodes and weights of three quadrature formulas with $N = 12$ internal nodes for $\alpha = 1/2$.

Gauss formula

Nodes	Weights
± 0.1205366802553231	0.2381498590107154
± 0.3546048870425356	0.2112734046606586
± 0.5680647467311558	0.1636775677557562
± 0.7485107481711011	0.1062659809389065
± 0.8854560256532099	0.0521909468652249
± 0.9709418174260520	0.0138404041661867

Gauss-Lobatto formula

Nodes	Weights
± 0.1125386591664354	0.2227082337053554
± 0.3319729790280478	0.2008200600303254
± 0.5347613738021686	0.1613787851381907
± 0.7107361043825368	0.1121957870878635
± 0.8510759734996212	0.0630108997211034
± 0.9487595481458028	0.0235582479049453
± 1	0.0017261498096647

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.1059475095439726	0.2099320804976957
± 0.3132148489095748	0.1917399646105878
± 0.5068030097762635	0.1585013510551252
± 0.6782654112077311	0.1159636387224015
± 0.8201485469120362	0.0714826655193443
± 0.9264326975154983	0.0327559472645929
± 1	k = 0 : 0.0050225157277009
	k = 1 : − (±) 0.0000475960425459

Table VI

The nodes and weights of three quadrature formulas with N = 40 internal nodes for $\alpha = -1/2$.

Gauss formula		Gauss-Lobatto formula	
Nodes	Weights	Nodes	Weights
± 0.0392598157590686	0.0785398163397448	± 0.0383027336900353	0.0766242110631657
± 0.1175373974578376	0.0785398163397448	± 0.1146834253984004	0.0766242110631657
± 0.1950903220161283	0.0785398163397448	± 0.1903911091646684	0.0766242110631657
± 0.2714404498650743	0.0785398163397448	± 0.2649815021966617	0.0766242110631657
± 0.3461170570774930	0.0785398163397448	± 0.3380168784085028	0.0766242110631657
± 0.4186597375374281	0.0785398163397448	± 0.4090686371713399	0.0766242110631657
± 0.4886212414969549	0.0785398163397448	± 0.4777198185122629	0.0766242110631657
± 0.5555702330196022	0.0785398163397448	± 0.5435675500012212	0.0766242110631657
± 0.6190939493098340	0.0785398163397448	± 0.6062254109666380	0.0766242110631657
± 0.6788007455329417	0.0785398163397448	± 0.6653257001655654	0.0766242110631657
± 0.7343225094356855	0.0785398163397448	± 0.7205215936007870	0.0766242110631657
± 0.7853169308807449	0.0785398163397448	± 0.7714891798219429	0.0766242110631657
± 0.8314696123025452	0.0785398163397448	± 0.8179293607667177	0.0766242110631657
± 0.8724960070727971	0.0785398163397448	± 0.8595696069872012	0.0766242110631657
± 0.9081431738250813	0.0785398163397448	± 0.8961655569610556	0.0766242110631657
± 0.9381913359224841	0.0785398163397448	± 0.9275024511020947	0.0766242110631657
± 0.9624552364536473	0.0785398163397448	± 0.9533963920549305	0.0766242110631657
± 0.9807852804032304	0.0785398163397448	± 0.9736954238777790	0.0766242110631657
± 0.9930684569549263	0.0785398163397448	± 0.9882804237803485	0.0766242110631657
± 0.9992290362407229	0.0785398163397449	± 0.9970658011837405	0.0766242110631657
		± 1	0.0383121055315828

Table VI (end)

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.0374124031244414	0.0748423121349573
± 0.1120279841070259	0.0748427919115832
± 0.1860170593675317	0.0748437733804742
± 0.2589658521376227	0.0748453025347435
± 0.3304664033547306	0.0748474541424904
± 0.4001188531487219	0.0748503398291132
± 0.4675336770402635	0.0748541211120416
± 0.5323338643489636	0.0748590301620624
± 0.5941570266352640	0.0748654032324979
± 0.6526574243965304	0.0748737358016783
± 0.7075079007053267	0.0748847766552883
± 0.7584017110201180	0.0748996953771060
± 0.8050542390282339	0.0749203964615829
± 0.8472045891365201	0.0749501472291872
± 0.8846170472282855	0.0749949368722048
± 0.9170824029742614	0.0750667325679930
± 0.9444191309862318	0.0751924094978149
± 0.9664744426304548	0.0754433775483860
± 0.9831253008699869	0.0760676581576409
± 0.9942802996426482	0.0785193613148634
± 1	k = 0 : 0.0673325708511871
	k = 1 : − (±) 0.0000318206856575

Table VII

The nodes and weights of three quadrature formulas with N = 40 internal nodes for $\alpha = -1/4$.

Gauss formula		Gauss-Lobatto formula	
Nodes	Weights	Nodes	Weights
± 0.0390138332726430	0.0780177540047503	± 0.0380742038561075	0.0761392057819651
± 0.1168039437900775	0.0777798378158427	± 0.1140018757576862	0.0759182166202100
± 0.1938828327096002	0.0773032680629220	± 0.1892686199447674	0.0754755884588661
± 0.2697811651106672	0.0765865265605418	± 0.2634380762463026	0.0748099852660685
± 0.3440367945175815	0.0756272233851904	± 0.3360802460241484	0.0739193088879052
± 0.4161975769259137	0.0744219496525588	± 0.4067739850905839	0.0728005766021706
± 0.4858241239265908	0.0729660588888508	± 0.4751094452894511	0.0714497403143189
± 0.5524924781658887	0.0712533526882093	± 0.5406904505840582	0.0698614285389648
± 0.6157966948540495	0.0692756320285230	± 0.6031367938734037	0.0680285815832173
± 0.6753513136104522	0.0670220521355493	± 0.6620864412153622	0.0659419331109137
± 0.7307937056075392	0.0644781784119802	± 0.7171976306672461	0.0635892623025323
± 0.7817862817493775	0.0616245680489366	± 0.7681508535543308	0.0609542898979232
± 0.8280185485009004	0.0584345628621813	± 0.8146507066339754	0.0580149972680588
± 0.8692089989991545	0.0548706957504199	± 0.8564276043108000	0.0547409632219120
± 0.9051068283320810	0.0508784889247362	± 0.8932393407050581	0.0510889257052385
± 0.9354934637660324	0.0463748994013989	± 0.9248724917423923	0.0469948872735867
± 0.9601839050745514	0.0412244211184213	± 0.9511436465532182	0.0423588063424663
± 0.9790278877377407	0.0351815552627065	± 0.9719004504305603	0.0370111292606514
± 0.9919110073207020	0.0277137759972869	± 0.9870223935788693	0.0306252308999533
± 0.9987579955162676	0.0171054337345859	± 0.9964207565849300	0.0224050543189941
		± 1	0.0060121230796757

Table VII (end)

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.0371993553920707	0.0743901769791779
± 0.1113924860562720	0.0741846624472427
± 0.1849700125037117	0.0737730651116982
± 0.2575253127897755	0.0731542183902699
± 0.3286574143353828	0.0723262948297118
± 0.3979732099308120	0.0712867056903047
± 0.4650896302914660	0.0700319536664111
± 0.5296357611720599	0.0685574245766333
± 0.5912548933612622	0.0668570961644862
± 0.6496064942672210	0.0649231301295544
± 0.7043680902724190	0.0627452939532572
± 0.7552370496019367	0.0603101260031802
± 0.8019322561601368	0.0575996991010703
± 0.8441956657788687	0.0545897302864977
± 0.8817937379795863	0.0512465764298055
± 0.9145187399986693	0.0475222307469186
± 0.9421899305713664	0.0433455283364940
± 0.9646546732414757	0.0386058531329399
± 0.9817897614591353	0.0331233808833753
± 0.9935054239606166	0.0266459566772971
± 1	k = 0 : 0.0129211311992661
	k = 1 : − (±) 0.0000085364830746

Table VIII

The nodes and weights of three quadrature formulas with N = 40 internal nodes for $\alpha = 0$.

Gauss formula		Gauss-Lobatto formula	
Nodes	Weights	Nodes	Weights
± 0.0387724175060508	0.0775059479784248	± 0.0378497165596036	0.0756632791071132
± 0.1160840706752552	0.0770398181642480	± 0.1133323498886541	0.0752298866132712
± 0.1926975807013711	0.0761103619006262	± 0.1881658256541514	0.0743655840531928
± 0.2681521850072537	0.0747231690579683	± 0.2619215043901503	0.0730753220622717
± 0.3419940908257585	0.0728865823958041	± 0.3341769201394594	0.0713664911253501
± 0.4137792043716050	0.0706116473912868	± 0.4045182002915358	0.0692488792419390
± 0.4830758016861787	0.0679120458152339	± 0.4725424361948476	0.0667346158568026
± 0.5494671250951282	0.0648040134566010	± 0.5378599909625933	0.0638381023757965
± 0.6125538896679802	0.0613062424929289	± 0.6000967312486392	0.0605759296625441
± 0.6719566846141795	0.0574397690993916	± 0.6588961702025955	0.0569667829837903
± 0.7273182551899271	0.0532278469839368	± 0.7139215093148015	0.0530313349381569
± 0.7783056514265194	0.0486958076350722	± 0.7648575674264752	0.0487921269603807
± 0.8246122308333117	0.0438709081856733	± 0.8114125857934221	0.0442734400302892
± 0.8659595032122595	0.0387821679744720	± 0.8533198987206503	0.0395011552017515
± 0.9020988069688743	0.0334601952825478	± 0.8903394598379915	0.0345026043991169
± 0.9328128082786765	0.0279370069800234	± 0.9222592142586162	0.0293064112161657
± 0.9579168192137917	0.0222458491941670	± 0.9488963054454345	0.0239423184107905
± 0.9772599499837743	0.0164210583819079	± 0.9700980966277576	0.0184409830446662
± 0.9907262386994570	0.0104982845311528	± 0.9857429295066996	0.0128336055583138
± 0.9982377097105592	0.0045212770985332	± 0.9957399700535163	0.0071497069724666
		± 1	0.0011614401858304

Table VIII (end)

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.0369899065376943	0.0739461237514734
± 0.1107676940982736	0.0735422567265864
± 0.1839405085400819	0.0727367285238125
± 0.2561087065136156	0.0715339387977439
± 0.3268781317188124	0.0699404569992117
± 0.3958622677561750	0.0679649865270904
± 0.4626843492931245	0.0656183172465939
± 0.5269794200427515	0.0629132666472972
± 0.5883963263618868	0.0598646099882175
± 0.6465996356636449	0.0564889998615867
± 0.7012714693225716	0.0528048757157233
± 0.7521132403678994	0.0488323640465265
± 0.7988472871123951	0.0445931702922962
± 0.8412183952208113	0.0401104642259296
± 0.8789952033760615	0.0354087627278062
± 0.9119714942636792	0.0305138205298110
± 0.9399673930498382	0.0254525651430269
± 0.9628305767010395	0.0202532351053826
± 0.9804380360205282	0.0149467091166233
± 0.9927028979393512	0.0095804103030450
± 1	k = 0 : 0.0029539377242157
	k = 1 : − (±) 0.0000024554760800

Table IX

The nodes and weights of three quadrature formulas with N = 40 internal nodes for $\alpha = 1/4$.

Gauss formula		Gauss-Lobatto formula	
Nodes	Weights	Nodes	Weights
± 0.0385354288733448	0.0770040672947052	± 0.0376291539992123	0.0751961517419049
± 0.1153773635551718	0.0763189391700099	± 0.1126744979251597	0.0745585307445338
± 0.1915338889816873	0.0749588614070951	± 0.1870821549132667	0.0732923115936444
± 0.2665525907016671	0.0729440767230862	± 0.2604310111154659	0.0714154433954979
± 0.3399878136152973	0.0703046664037247	± 0.3323059450128303	0.0689546106398807
± 0.4114033094333004	0.0670802190597029	± 0.4023001768126810	0.0659449526859565
± 0.4803748282566034	0.0633194028842813	± 0.4700175706338209	0.0624297008648011
± 0.5464926388827976	0.0590794542711265	± 0.5350748764495800	0.0584597434231666
± 0.6093639628703553	0.0544256003257211	± 0.5971038990986031	0.0540931321615393
± 0.6686153079064700	0.0494304390302499	± 0.6557535820847316	0.0493945493636957
± 0.7238946866278614	0.0441733097725126	± 0.7106919943669093	0.0444347602805652
± 0.7748737077357944	0.0387397008176844	± 0.7616082088834574	0.0392900864882667
± 0.8212495270317590	0.0332207634005561	± 0.8082140621552476	0.0340419516917452
± 0.8627466468992521	0.0277130434400715	± 0.8502457849551940	0.0287765796239840
± 0.8991185538436213	0.0223186215921287	± 0.8874654946820113	0.0235849756509992
± 0.9301491852530141	0.0171460199503888	± 0.9196625406267166	0.0185634274064632
± 0.9556542197605615	0.0123126257456460	± 0.9466546933696572	0.0138149862198342
± 0.9754821970204613	0.0079504408298061	± 0.9682891672471698	0.0094529447953929
± 0.9895155433287959	0.0042204994331293	± 0.9844434490844750	0.0056089009023429
± 0.9976724619263318	0.0013584332124137	± 0.9950257836232911	0.0024534613972924
		± 1	0.0002579836925329

Table IX (end)

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.0367839563612589	0.0735099148970169
± 0.1101533106245896	0.0729149877202420
± 0.1829280613849857	0.0717331598560625
± 0.2547153734437443	0.0699804022047699
± 0.3251277421137847	0.0676804677572417
± 0.3937850851556351	0.0648646527762240
± 0.4603167947209818	0.0615714872869307
± 0.5243637382760115	0.0578463630306950
± 0.5855801977857772	0.0537411098585129
± 0.6436357368376520	0.0493135351637014
± 0.6982169858960027	0.0446269459274808
± 0.7490293365807635	0.0397496802624053
± 0.7957985369212621	0.0347546868113569
± 0.8382721814093951	0.0297192095439129
± 0.8762210936873601	0.0247246697009773
± 0.9094406101659659	0.0198569020605205
± 0.9377518060604118	0.0152070386513246
± 0.9610028364382858	0.0108736484594080
± 0.9790712606650266	0.0069676143174637
± 0.9918743228540257	0.0036249500999967
± 1	k = 0 : 0.0007577583777963
	k = 1 : – (±) 0.0000007496620279

Table X

The nodes and weights of three quadrature formulas with N = 40 internal nodes for $\alpha = 1/2$.

Gauss formula		Gauss-Lobatto formula	
Nodes	Weights	Nodes	Weights
± 0.0383027336900353	0.0765117957284665	± 0.0374124031244414	0.0747375562556892
± 0.1146834253984004	0.0756164273668076	± 0.1120279841070259	0.0739034947236856
± 0.1903911091646684	0.0738466773190081	± 0.1860170593675317	0.0722540032098954
± 0.2649815021966617	0.0712440270262177	± 0.2589658521376227	0.0698259286162079
± 0.3380168784085028	0.0678694804053954	± 0.3304664033547306	0.0666735100962048
± 0.4090686371713399	0.0638021339692002	± 0.4001188531487219	0.0628671674402566
± 0.4777198185122629	0.0591373228707736	± 0.4675336770402635	0.0584919280108434
± 0.5435675500012212	0.0539843863285520	± 0.5323338643489636	0.0536455273671861
± 0.6062254109666380	0.0484641048076587	± 0.5941570266352640	0.0484362260070804
± 0.6653257001655654	0.0427058690281698	± 0.6526574243965304	0.0429803909944708
± 0.7205215936007870	0.0368446471563021	± 0.7075079007053267	0.0373998964917416
± 0.7714891798219429	0.0310178212649910	± 0.7584017110201180	0.0318194012576767
± 0.8179293607667177	0.0253619672145401	± 0.8050542390282339	0.0263635639128661
± 0.8595696069872012	0.0200096534302124	± 0.8472045891365201	0.0211542581463555
± 0.8961655569610556	0.0150863336107018	± 0.8846170472282855	0.0163078499907762
± 0.9275024511020947	0.0107074061997652	± 0.9170824029742614	0.0119325977648771
± 0.9533963920549305	0.0069755095445083	± 0.9444191309862318	0.0081262320534607
± 0.9736954238777790	0.0039781161395212	± 0.9664744426304548	0.0049737669995729
± 0.9882804237803485	0.0017854823457440	± 0.9831253008699869	0.0025455770080639
± 0.9970658011837405	0.0004490016409127	± 0.9942802996426482	0.0008956456792225
		± 1	0.0000636413713149

Table X (end)

Generalized Gauss formula (m = 2)

Nodes	Weights
± 0.0365814085255955	0.0730813225088181
± 0.1095490494944641	0.0723022976674439
± 0.1819322036588432	0.0707608515469097
± 0.2533446790788294	0.0684898373955743
± 0.3234054633329635	0.0655376579380642
± 0.3917407566886092	0.0619672337553654
± 0.4579859669068639	0.0578546622421085
± 0.5217876551156113	0.0532875957231149
± 0.5828054224979983	0.0483633732969717
± 0.6407137279585556	0.0431869462234237
± 0.6952036274917380	0.0378686410719452
± 0.7459844267951954	0.0325218083074362
± 0.7927852400054320	0.0272604064326174
± 0.8353564499781472	0.0221965731855422
± 0.8734710712795985	0.0174382355831728
± 0.9069260324156574	0.0130868098331731
± 0.9355434427657936	0.0092350404714158
± 0.9591721013998029	0.0059650264704625
± 0.9776905057172086	0.0033464886781565
± 0.9910211789395672	0.0014354926651214
± 1	k = 0 : 0.0002118624006111
	k = 1 : − (±) 0.0000002410658004

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16. Abstract

We propose some spectral type collocation methods well suited for the approximation of fourth-order systems. Our model problem is the biharmonic equation, in one dimensional and in two dimensions when the boundary conditions are periodic in one direction. It is proved that the standard Gauss-Lobatto nodes are not the best choice for the collocation points. Then, we propose a new set of nodes related to some generalized Gauss type quadrature formulas. We provide a complete analysis of these formulas including some new issues about the asymptotic behaviour of the weights and we apply these results to the analysis of the collocation method.

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